INTERRELATED DEMAND AND SUPPLY IN THE MARKETS FOR NEW AND USED DURABLE GOODS

By

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A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN
PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF PHILOSOPHY

UNIVERSITY OF FLORIDA

1986

ACKNOWLEDGMENTS

I am deeply indebted to my supervisor Dr. Edward Zabel for introducing me to the techniques used in this study and for his numerous suggestions and corrections. Without his help this dissertation would not have been possible.

Dr. Stephen Cosslett, with his sharp observations during my proposal presentation, benefited me in determining the direction of this study. Special thanks are also due to the other members of my Ph.D. committee, Drs. William Bomberger and John Shonkwiler.

Members of the Economic Departments in the University of Florida and Radford University provided useful comments during my seminar presentations.

Finally, I am indebted to my wife Gilda for spiritual support and assistance with preliminary typing and to Ms. Sandy Ostrofsky for her professional typing.

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of the University of Florida in Partial Fulfillment of the
Requirements for the Degree of Philosophy

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August 1986

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The purpose of this study is to analyze the behavior of the consumers and suppliers in a durable goods market. The producer of the durable good, which lasts two periods, is assumed to be a monopolist. After the good has been used for one period by the consumer, it can be traded in the Second Hand Market (SHM). Thus, the monopolist faces competition from the SHM. In the existing literature, in which perfect certainty is assumed, the interrelation between the new goods market and the SHM is recognized but does not appear explicitly in the formal models. Introduction of a more realistic framework in which uncertainty about the future is recognized and price/quantity setting by the suppliers is assumed, results in optimal strategies for both markets which are explicitly interrelated. In other words, state variables and parameters of the SHM structure will explicitly enter the new goods monopolist's optimal strategy formulas and vice versa.

The introduction of uncertainty complicates considerably the analysis of the consumer's behavior. The purchase of a new durable good

is a joint consumption-investment decision. The consumer consumes the good's services in the current period and invests in the remaining services. Under certainty, the investment is riskless because all future prices are known. Under uncertainty the investment in the durable good becomes risky and this gives rise to a non-trivial portfolio problem. In this study the joint consumption-investment problem of the consumers is analyzed and demand functions for new and used goods are derived. These functions depend only upon current prices. The properties of these functions are also derived.

The competition between suppliers of new and used goods gives rise to game-theoretic situations. Two cases are analyzed, the non-cooperative and the cooperative case. The optimal strategies are derived for both cases. Also, partial adjustment mechanisms, showing that the system adjusts towards its endogenously determined equilibrium, and stationary distributions for the state and choice variables of the system are derived.

CHAPTER ONE

INTRODUCTION

The stimulus for this study was motivated by two themes concerning durable goods. The first involves the persistent reference in the literature to the importance of a Second Hand Market (SHM), along with the absence of studies examining the impact of this market upon the behavior of monopolistic producers of durable goods. This theme is developed at length in Chapter Two. The second involves my conviction that price and quantity setting behavior in the face of uncertainty is widespread in the economy, especially in the case of durable goods, and this behavior is manifested through the existence of inventories and sometimes order backlogging.

In the following chapters, I use the price and quantity setting model, used in the theory that examines the behavior of monopoly under uncertainty, to examine the behavior of a durable goods monopolist. The introduction of uncertainty exposes the interdependence of the markets for new and durable goods, and shows how the structure of the Second Hand Market will affect the behavior of the monopolist.

The introduction of uncertainty about the future, while a worthwhile step toward a more realistic model, proved a difficult task. Many of the assumptions of the certainty models have to be reconsidered. In particular, the assumption that competitive forces in the market will lead to a new goods price equal to the discounted stream of the prices

of the services of the good during its lifetime, is no longer valid due to the uncertainty about future prices. Consequently, the demand functions used in the certainty models are no longer acceptable under uncertainty. Thus, the first task of the study became the analysis of consumer behavior under uncertainty in the quest for demand functions consistent with uncertainty. The analysis of the demand side brought to surface a very interesting characteristic. The purchase of a durable good is a joint consumption-investment decision. The consumer consumes the services of the good in the current period and simultaneously invests in a risky asset which provides services during the remaining lifetime of the good. Risk arises since future prices are uncertain.

As it is shown in Chapter Three, the analysis of the consumer's behavior leads to demand and excess demand functions of current prices only.

The next step of the analysis is the examination of the behavior of the suppliers in the markets for new and used goods. The complexity of the problem made inevitable the introduction of a series of simplifications in order to reduce the analysis to a manageable level. First, the demand and excess demand functions, already derived under the assumption of zero probability of rationing, were too complicated to be used in their exact form. Thus, I use linear approximations of these functions, retaining their properties (signs). Second, the durable good is assumed to last only two periods. This assumption reduces the number of prices in the market to two. Third, the cost of production of durable goods is assumed linear. This assumption is probably the most crucial because it implies that the optimal strategies in both markets are independent of the new goods inventory. Fourth, it is assumed that one dealer sets

the price for all the dealers in the SHM. This last assumption leads to a duopoly situation in the durable market system. There are two sellers, the monopolist and the SHM dealer, who sell goods whose services are perfect substitutes. Thus, game situations develop in the durable goods market system. The non-cooperative (Nash) equilibrium solution is examined in Chapter Four, and the cooperative equilibrium solution (and how it compares to the non-cooperative case) is examined in Chapter Five. By a cooperative solution is meant that the new goods producer also controls the SHM.

Despite various simplifications, the results which are summarized in Chapter Six provide a first step in a realistic study of the interaction between new and used goods markets. Finally, further research in the topic is warranted and numerous suggestions for extensions of research are offerred in Chapter Six.

CHAPTER TWO

REVIEW OF THE LITERATURE

The development of the literature analyzing the durable goods market revolves mainly around the following questions:

(a) What is the effect of the market structure upon the durability of the product?

(b) Does the monopolist lose his market power when the good is infinitely durable and all transactions take place instantaneously? What are the devices the monopolist can use to prevent his loss of power?

(c) What are the effects of the existence of a SHM upon the pricing and production behavior of the producer of new goods?

In the next three sections I analyze the answers given to these questions by various authors, pointing out their limitations. A fourth section reviews the literature on monopoly behavior under uncertainty.

Durability and Market Structure

The analysis of the effects that different market structures have upon the durability of the product was the main theme of the literature examining the durable goods market during the 1960s and early 1970s. Schmalensee (16) reviews all this literature. As he points out, all the economists analyzing the market for durable goods during the 1960s were in agreement that a monopolist will produce fewer and less durable goods than a perfect competitor. Schmalensee (16) cites papers by Martin,

Kleiman and Ophir, Levhari and Srinivasan, and himself that reached the above conclusion. During the early 1970s, in a pair of papers, P. Swan (19,20) overturns the prevailing conclusion by showing, under the same assumptions, its falseness. In particular, Swan proved that as long as the good is demanded for the services it renders and not for its durability *per se*, and given that there are no economies of scale in the production of durability, the market structure will have no effect upon the durability of the good. In other words, a monopolist will produce goods equally durable to those of a competitive firm, albeit fewer in amount. The optimal durability built in the good will be determined by purely technological factors.

Schmalensee (16) spells out all the assumptions necessary to derive Swan's "independence" result and reviews subsequent attempts to examine the effects of relaxing these assumptions. He concludes that in most cases the relaxation of the assumptions does not lead to clear-cut conclusions about the validity of the "independence" result.

The research in the area of the relation between market structure and durability is still quite active. Liebowitz (11) shows how the introduction of demand for durability *per se*, i.e., postulating that the demand for new goods is much larger that the demand for used goods (which in turn means that their services are not perfect substitutes), combined with the existence of a SHM will result in a less durable good when the producer is a monopolist.

Bulow (6) argues that Swan's "independence" result holds for a monopolist renter, but not a monopolist seller who will produce a less durable good. Bulow derives this result from an arithmetic example whose general validity is questionable.

Intertemporal Pricing of Durable Goods

The set of questions concerning the loss of market power of a monopolist was set and answered by R. Coase (7). In this well-known paper, Coase analyzes the pricing problem that a durable goods monopolist faces. He assumes a linear time-stationary demand for an infinitely durable good produced under constant marginal cost. The argument goes as follows: Initially the monopolist, in order to maximize his profits, will be willing to set the price at the level determined by the usual "Marginal Cost Equals Marginal Revenue" condition, and sell the corresponding quantity. After the transactions take place, the monopolist will see that he may still make some additional profit, given that there will still exist unsatisfied demanders willing to pay a price higher than the marginal cost. Therefore, the monopolist will have the incentive to sell additional units of the good up to the point where the price will become equal to the marginal cost. If now we assume a perfect certainty environment where the consumers are aware of the monopolist's incentive to gradually lower the price in order to realize additional profits, and if all the transactions take very little time, then no consumer will be willing to pay a price above marginal cost. As a result, the monopolist will lose his market power. Furthermore, under the above assumptions, a durable goods monopolist is as efficient as a perfect competitor! Coase's result looked shocking, and, given that his analysis was informal and largely intuitive, there were subsequent attempts to see if the same results hold in a formal context.

N. Stokey (17), using a formal model, reveals that Coase's result is valid in a continuous time framework. Moreover, she shows that, when the time framework is discrete, the monopolist will be able to price

discriminate overtime by traveling down the time-stationary demand curve. Unfortunately, Stokey's analysis suffers from two major drawbacks. First, she fails to explain the source of uncertainty in her model that will require the formation of rational expectations from the consumer's side. Thus, she ends up using expectations in a perfect certainty framework. Second, her definition of rational expectations seems to differ substantially from the conventional definition, i.e., while the conventional definition requires that consumers form their expectations by exploiting all available information, Stokey constrains the consumers' information set to contain only knowledge of the outstanding stock of durable goods. Thus, when a sudden change in the outstanding stock occurs, the consumers do not take into account the probability of this change happening again or the fact that this change will modify the monopolist's plans. Therefore, due to the constrained information set the consumers end up with consistently wrong expectations. These drawbacks cast shadows upon the validity of Stokey's discussion concerning the non-uniqueness of Rational Expectation Equilibrium (REE) versus the uniqueness of the Perfect Rational Expectation Equilibrium (PREE), but I do not think that affects Coase's results.

Coase (7) proposes different devices the monopolist can use to retain all or part of his market power. These devices include: (i) renting instead of selling the durable good; (ii) writing contracts that will insure the consumers against future changes in prices and quantities; (iii) agreements to buy back the product at a price slightly below the purchase price, and (iv) making the product less durable.

Coase's proposals are formally analyzed in the subsequent literature. Bulow (6) explicitly compares a monopolist renter and monopolist seller and shows that in a two period horizon model, renter's profits are considerably higher than seller's.

V. Suslow (18) examines the behavior of a monopolist who makes a binding commitment to keep the price constant in all future periods. Under this commitment the monopolist will produce the static monopoly quantity in the first period and nothing afterwards. P. Swan (21), in a paper examining the aluminum industry in the interwar period, derives the same result in the case when aluminum recovery is complete (i.e., the good is infinitely durable).

Stokey (17), Suslow (18) and Bulow (6) examine a different type of "commitment" which is endogenous in the discrete time models. Specifically, while the monopolist does not commit himself to maintain a particular price or quantity path, the use of a discrete time framework implies that during each period the price and quantity will be given. In other words, the monopolist "commits" himself to produce a specific amount in the beginning of the period, sell it at a particular price and then wait till the beginning of the next period to produce again. This form of "limited" commitment enables the monopolist to price-discriminate intertemporally by exploiting the eagerness of the consumers who are not willing to wait for one period in order to enjoy a lower price. The common result of the discrete time models is that the monopolist will indeed undertake intertemporal price discrimination, by travelling down the demand curve, in order to obtain part of the consumers' surplus. The determination of the price that the monopolist will charge each period presents a major problem. The difficulties arise from the fact

that the demand for the durable good is a function of the discounted stream of the price of services the good will provide in its future life. In other words, current demand is a function of current and future prices of services. This makes the solution of an infinite horizon problem very difficult. As a consequence, the researchers have to assume a finite horizon after which the price of services will become zero. The appropriateness of the finite horizon model in analyzing the monopolist's pricing behavior is questionable. One of the problems with the finite horizon models is that we are not able to determine what the steady-state of the system is. Another problem is the counterintuitive result that at the beginning of the horizon the monopolist travels down the demand curve very slowly and as the horizon approaches its end, he lowers his price faster. This means that as the horizon expires the price becomes more volatile. However, intuition suggests the opposite, i.e., at the beginning of the horizon we will observe large decreases in price. As time passes and the system approaches its steady-state (which as Coase showed is the competitive equilibrium, given infinite durability), the price will change only marginally approaching the competitive price.

The reason why the finite model reaches counterintuitive results lies in the fact that after the end of the horizon the value of the goods for the consumers goes to zero. Therefore, goods purchased in the last period will last only one period, goods purchased during the next to the last period will last for two periods and so on. Consequently, in order to sell them, the monopolist will have to reduce the price dramatically. Specifically, as Suslow (18) shows, if the horizon consists of T periods, where T is the last period, then it holds that $P_T = \frac{1}{2}P_{T-1}$, i.e., the price in the last period will be half of the price

the period before. The same result, i.e., the dramatic price decrease, is obtained by Bulow (6) and I suspect that many of Bulow's surprising results depend heavily upon the finite horizon framework. More specifically, the lower price in the last period will result in lower profits. This downward bias in profits may be responsible, for example, for Bulow's contention that a monopolist seller will prefer a lower fixed cost-higher (but constant) marginal cost technology compared with a monopolist renter.

Swan (21) examines Coase's fourth device, i.e., reducing the durability of the product, in the context of the aluminum industry. In particular Swan examines the effects of the existence of a competitive aluminum recycling sector upon the behavior of a monopolist producer of a virgin aluminum (ALCOA). Assuming different aluminum recovery rates which correspond to different depreciation rates, Swan derives the market's steady-state equilibrium. When the monopolist is not commited with respect to price and the recovery of aluminum (by scavangers) is complete (i.e., depreciation zero or infinite durability), then the monopolist, in the steady-state, loses all his ability to make profits and ends up producing the efficient quantity (i.e., the quantity at which the price equals marginal cost). If now aluminum recovery is partial, i.e., depreciation is partial, then the monopolist, in the steady-state, will be producing each period in order to replenish part of the depreciated stock and thus he will be able to charge a price above marginal cost. Finally, if aluminum recovery is zero, i.e., aluminum lasts only one period, then the monopolist will produce his profit maximizing quantity each period and charge the static monopoly price (which corresponds to the monopolist renter's case). Swan also

shows that the above results, i.e., that the monopolist's profits increase as durability decreases (depreciation increases), hold for the case where the monopolist makes a price commitment to his customers.

Suslow (18) derives the same result as Swan, i.e., showing that with a constant rate of depreciation, the monopolist will charge a higher price (and will produce a lower stock) than with no depreciation. The explanation for this result lies in the fact that the monopolist now retains his monopoly power over the part of the product that depreciates.

Another reason for which limiting the durability will increase the monopolist's profit is that the monopolist's incentive to lower the price, and thus impose capital loses on the customers that already purchased the good at a high price, is not effective anymore because the monopolist will have to face the same consumers in the future when their goods expire and they return to the market to buy another unit. The loss (or weakening) of the above incentive will be known to the consumers who will now be willing to pay a price higher than the marginal cost. Therefore, in cases where the durability of the good is finite the monopolist will retain part of his monopoly power.

Effects from the Existence of Second Hand Markets

In 1974 we witnessed the simultaneous appearance of two independently produced papers which analyze the working and the effects of the SHM and which arrive at almost identical results. H.L. Miller (13) and D.K. Benjamin and R.C. Kormendi (2) examine the relationship between the markets for new and used goods. Assuming that the durable good lasts two periods and that the marginal production cost is increasing, both

papers show that, when the structure of the market is competitive, the producers of the new goods are better off by banning the SHM when their marginal cost is low and by supporting the SHM when their marginal cost is high.

The reason is that a low marginal cost implies a low price and as a result producers' suplus will be maximized if a large number of units is sold. Thus, producers of new goods will wish to ban the SHM in order to increase sales. From the other side, when marginal cost is high, the price will be high and the producers will maximize the producers' surplus if there is a SHM where consumers can realize a resale price, which in turn enables them to pay the high price for the new product. Under a monopolistic structure the above result is no longer valid. What becomes crucial now is the degree of substitutability between new and used goods. If the new and used goods are perfect substitutes, in other words if there is no demand for durability per se, then the consumers will be indifferent between services coming from new or used goods and the demand for new goods will be equal to the demand for used goods. Under these circumstances, H.L. Miller (13) shows that a monopolist who operates under constant marginal cost conditions will always support the existence of a SHM. In the limiting case where marginal cost is zero, the monopolist will be indifferent with respect to the existence of a SHM. If now there exists a strong consumers' preference for durability _per se_, the demand for new goods' services will be much higher than the demand for used goods' services. In this case the results derived for the competitive industry apply also for the monopolist. Specifically, Liebowitz (11) shows that a monopolist with low marginal cost will be willing to kill the SHM, while Benjamin and

Kormendi (2) show that a monopolist with high marginal cost will be willing not only to support the existence of a SHM but also, in the case where the demand for used goods is small and the marginal revenue becomes negative quickly, to intervene and support a high price for the used goods by buying some of them. The high resale price of the used goods will induce the consumers to pay a higher price for the new good. The support of the price of the used goods was one of the devices that Coase (7) proposed for the durable goods monopolist to increase profits.

Swan (21) compares ALCOA's profits with and without the existence of a SHM. In his first two models (examined in the previous section) Swan assumes that there is no SHM in which the consumers realize a resale value for their used aluminum. Instead the consumers discard their aluminum which is recovered by scavangers. In his third model, he introduces a SHM. Assuming that the monopolist precommits to keep the price of virgin aluminum constant, Swan shows that if marginal cost is constant and the rate of aluminum recovery (depreciation) is constant, then both ALCOA and consumers will benefit from a low cost recycling sector.

From the other side, with an endogenous rate of recovery and given that the monopolist following the marginal cost equals marginal revenue rule sets the price higher than marginal cost while the competitive recylcing sector sets price equal to marginal cost, equalization of prices will lead to overproduction in the recycling sector (i.e., the marginal cost in the recycling sector will be higher than the cost of the marginal unit of virgin aluminum). Swan does not give a comparison of the monopolist's profits with and without the SHM. He is mainly concerned with the effects upon the consumer and argues that integration

of the recycling section under ALCOA will increase the social welfare by substituting low marginal cost virgin aluminum for high marginal cost recycled aluminum. His empirical analysis seems to support this conclusion. A counterargument, that applies only in the case of non-renewable resources such as aluminum, is that before we draw conclusions about social welfare we should include the opportunity cost of exhausting the non-renewable resource. Inclusion of such a cost may reverse Swan's result by making the high cost recovered aluminum socially desirable in the face of diminishing bauxite reserves.

Firm Behavior Under Uncertainty

In this section I describe the models analyzing monopolistic or oligopolistic firms' behavior under uncertainty developed by Zabel (23,24,27), Thowsen (22), Reagan (15), Blinder (3), and C. Miller (12). The common characteristic of all these models is the use of dynamic programming techniques to determine the firm's optimal strategy. Dynamic programming techniques were developed in the early 1950's and one of their first applications was in inventory problems. The dynamic programming framework facilitates the analysis of the behavior of a price and quantity setting firm in a stochastic environment. If a firm sets both price and quantity in the face of a stochastic demand, this procedure means that the market does not clear except by chance. In other words, dynamic programming enables us to study non-market clearing situations.

The common assumption of the above papers is that the uncertainty enters the model as an additive or multiplicative random term included in the demand function. Under the assumption that the firm announces its price and quantity supplied before the realization of the random

demand, the possibility of ending up with excess demand or supply at the end of the trading period arises. The excess supply is kept as inventory while the treatment of the excess demand divides the models in two categories. One category of models, called "lost sales" models, assumes that the excess demand is lost, while the second category, the "backlogging" models, assumes that the firm can carry the excess demand to the next period as unfilled orders, in other words, the excess demand is treated as negative inventory. Due to the symmetry resulting from the possibility of negative inventories, backlogging models are relatively easier to solve.

Also, by specifying the form of the demand and cost functions, exact expressions for the firm's optimal strategy can be derived (see Zabel (24), Blinder (3)). Zabel (24) examines the lost sales case using a one period and a finite period horizon. He proves the existence and, under some conditions concerning the distribution of the random term, the uniqueness of optimal strategy. Thowsen (22) shows the relationship between lost sales and backlogging models.

The specific forms of the demand and cost functions are crucial for the results of the formentioned papers. Due to the considerable complexity of the developed models, most of the writers use linear forms for some or all functions. But Zabel (24) reveals that the existence of a unique optimal solution can be proved under the general assumption of a concave demand function and convex production and inventory-holding cost functions. (Specifically, Zabel (24) used a linear inventory-holding cost function, but the extension to a convex form is not expected to alter the results.)

In the backlogging models, where exact expressions of the optimal strategy and the general form of the value function can be derived, the

specification of the production cost function is crucial. The value function inherits the form of the production cost function, assuming that the inventory and backlogging costs are convex.

Under linear production costs the resulting optimal strategy is a constant price-constant optimal stock policy. In other words, the firm will charge a single price and will replenish its inventory up to an optimal level each period, except in a transitional period where the inventory may be above its optimal level, in which case production ceases and the firm charges a lower price in order to reduce its excessive inventory. Under quadratic production and inventory costs, the optimal price and supply policy will depend upon the level of inventory. Furthermore, partial adjustment mechanisms for inventory, price, production and supply are revealed under quadratic costs [see Zabel (27)].

C. Miller (12) extended the previous analysis to a duopoly situation. He showed that the introduction of a second firm in the market complicates the problem considerably. Nevertheless, he was able to derive the optimal price and supply policy for each duopolist when the system is in a non-cooperative (Nash) equilibrium. To derive his results Miller assumed linear demand and cost functions in a backlogging model. As was to be expected, he derived a constant price-constant stock policy for each duopolist.

One of the drawbacks of the dynamic programming techniques used in the above models is that their level of difficulty increases rapidly when additional choice and state variables are considered. This makes very difficult the use of the technique to examine general equilibrium models.

Another drawback, which is not unique in the reviewed models, is the fact that the analysis is essentially static. Even if the examined models are optimizations over some finite or infinite horizon, they assume that the structure of the model remains invariable during that time, which is hardly the case in the real world. Dynamic analysis is needed to show how firms choose prices and quantities in the face of an ever changing world.

CHAPTER THREE

DEMAND FOR NEW AND USED DURABLE GOODS UNDER UNCERTAINTY

Introduction

The purpose of this chapter is to study the behavior of a representative consumer in the market for new and used durable goods under the assumption that future prices are unknown. The result of this study will be the derivation of demand functions for new and used goods and the analysis of their properties.

The existing literature examines the consumer's behavior under the certainty assumption, i.e., it assumes that the consumer knows all current and future prices [Parks (14)]. Assuming that the consumer demands the durable good for the services it provides, Parks uses a continuous time model, in which the utility is maximized over an infinite horizon, to derive a demand for the services of the good as a function of their price (rental rate). From the first order conditions of the problem, Parks also derives a relation between the price of a (new or used) good and the price of its services (rental rate). In particular, the relation between price and rental rate indicates that the price of the durable good must be equal to the present value of its services over the rest of its life. Combining the relation between price and rental rate with the demand for services, the demand for durable goods of any age, as a function of its current and future services prices, can be derived. Bulow (6) uses such a demand function in his model.

Introduction of uncertainty about future prices destroys the simplicity of the relationship between the price of the durable good and the present and future rental rates. The consumer now has to form expectations about future prices and rental rates and, therefore, no exact relations are possible anymore. Furthermore, I will introduce another assumption consistent with the analysis in the following chapters, that is, the prices in the market are set in the beginning of each period by price makers who are not constrained to clear the market. This assumption gives rise to the possibility of rationing and therefore the consumer will have to take into account the probability of being rationed in maximizing his expected utility.

Because of the complexity of the model, which will become evident in the following sections, in the main text I examine the limiting case where the probabilities of rationing are zero. In Appendix A, I make an effort to introduce positive probabilities of rationing.

In what follows, I first describe the structure of the model, then examine the utility maximization problem with only one period remaining in the horizon. Subsequently, I expand the horizon to two periods and derive specific expressions for the demand functions. In a fourth section, I examine the properties (signs) of the derived demand functions. In the last section, a summary and extension of the derived results is given. In the Appendix I introduce positive probabilities of rationing and show that the demand functions in this case are still functions of the same variables as the one in the main text.

The Structure of the Model

One of the basic assumptions of the model is that the consumer derives utility by consuming the services of a durable good and not from

the durability per se. In other words, the consumer does not care if the services come from a new or an old good. Assume now that the consumer derives utility by consuming two goods, a perishable good (called good A) and the services of a durable good. The durable good lasts two periods and has no salvage value after that. The units of the good are defined such that one unit of the good gives one unit of services each period of its life. The durable good is produced by a monopolist and traded during the first period of its life in the new goods market. The new durable good will be called B. In the new goods market the monopolist will set its price at the beginning of each trading period. For the purposes of this chapter, I will assume that the monopolist's supply of new goods is high enough to preclude any excess demand (i.e., probability of rationing zero). During the second period of its life the durable good can be traded in the SHM. The used durable good will be called C. The SHM is operated by only one dealer (or many dealers acting collectively) who sets a price* at which he buys and sells the used goods. For simplicity I assume that the spread between the buying and the selling price is zero. For a positive spread, and how its optimal level is determined, see Zabel (26). Here also I assume that the SHM dealer has enough inventory on hand so that all consumers' orders are satisfied.

Each period the consumer has two options; he can buy a new durable good from its monopolist producer or he can buy or sell a used durable good in the SHM. The consumer's utility function is assumed to be a Cobb-Douglas function. The specific form of the function is chosen to

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*(Which in the present setting is the same as the rental rate.)

facilitate exact calculations. In principle, any concave function with regular properties should give the same general forms and properties for the demand functions that will be derived. The specific form of the utility function is

$$U = z_A^{(1-\gamma)\lambda} \; z_S^{\gamma\lambda}, \; 0 < \gamma < 1, \; 0 < \lambda < 1.$$

where z_A is the quantity consumed of the good A and z_S is the quantity consumed of the services of new and used durable goods. The above utility function satisfies Stiglitz's condition for constant relative risk aversion and $(1-\lambda)$ is the measure of relative risk aversion (Zabel (25)). Given the definition of the units of the durable good and letting z_B be the quantity of new durable goods, v_C the consumer's inventory of used goods, and z_C the adjustment in the quantity of the used goods, the following equation holds:

$$z_S = z_B + z_C + v_C.$$

Note that the quantity of new goods z_B that the consumer buys cannot be negative, while the adjustment in the quantity of used goods z_C can be negative up to $-v_C$, i.e., the consumer can sell all his used goods inventory. The used goods' inventory v_C equals the quantity of new goods bought last period, which after one period's use became used goods. Therefore, v_C cannot be negative. The following must always be satisfied:

$$z_S \geq z_B, \; z_B \geq 0.$$

The following analysis of the consumer's behavior follows closely methods used by Hakansson (9) and Zabel (25) in treating similar problems. As in these papers, I assume that the consumer knows his income stream during the horizon of the problem with certainty, or, what amounts to the same thing, the consumer receives a certain amount of

wealth w at the beginning of the horizon and nothing thereom (except his portfolio returns). The initial wealth consists of an amount of cash plus the value of the inventory of used goods at the beginning of the horizon. The consumer consumes part of his wealth each period and invests the remainder. The consumer's portfolio consists of a riskless one-period bond whose return is R percent per dollar and of the risky durable good. The riskiness of the durable good stems from the uncertainty about next period's prices. Thus, when the consumer buys a new durable good, he makes a joint decision. He decides to consume the services of the good for one period evaluating them at the current rental rate (the price of a used good) and to invest the difference between the price of the new good and the rental rate. Comparison of this difference with next period's rental rate will give the return of the investment on the risky durable good. The return of the riskless asset per dollar invested is r (r = 1 + R).

The consumer is assumed to know the current prices p_A, p_B, p_C of the three goods and to form probabilistic expectations about future prices.

Assume now that the consumer maximizes his expected utility over a finite horizon. Let t be the number of periods remaining in the horizon. The consumer's value function satisfies the equation:

$$f_t(w_t;p_t,v_{Ct}) = \max_{z_{At},z_{Bt},z_{St}} \{z_{At}^{(1-\gamma)\lambda} z_{St}^{\gamma\lambda} + \alpha E_{p_{t-1}} f_{t-1}(w_{t-1};p_{t-1};z_{Bt})\}$$

subject to:

$$w_t \geq p_{At}z_{At} + (p_{Bt} - p_{Ct})z_{Bt} + p_{Ct}z_{St}$$

$$z_{St} \geq z_{Bt}$$

$$(z_{At},z_{Bt}) \geq 0$$

and $f_0 = 0$,

where $p_t = (p_{At}, p_{Bt}, p_{Ct})$, i.e., the price vector during period t. The expectation operator is over next period's price vector p_{t-1} (note that next period only t-1 periods remain in the horizon) and α is the utility discount factor with $0 < \alpha < 1$. The wealth constraint is derived as follows. Initial wealth is equal to cash plus the value of the initial inventory $p_{Ct}v_{Ct}$. After the consumer makes his purchasing decisions and neglecting the purchase of bonds, $w_t \geq p_{At}z_{At} + p_{Bt}z_{Bt} + p_{Ct}z_{Ct} + p_{Ct}v_{Ct}$. Strict inequality holds if the consumer buys a positive amount of bonds, while equality implies zero quantity of bonds purchased. As we saw previously, the amount of services the consumer consumes is equal to the units of the durable good he possesses, i.e., $z_S = z_B + z_C + v_C$. Therefore, it follows that $p_C z_S = p_C z_B - p_C z_C + p_C p_C$. Substituting this outcome in the above inequality yields the wealth constraint. The derivation of the other two constraints has already been discussed above.

The One-Period Horizon Problem

In this section I assume that only one period remains in the horizon and examine the consumer's optimal orders (consumption). With only one period remaining, no rational consumer will invest in bonds, given that any wealth remaining in the end of the horizon is assumed to yield no utility ($f_0 = 0$). Therefore, the wealth constraint will hold with equality. The functional form of the one-period value function will be:

$$f_1(w_1, p_1, v_{C1}) = \max_{z_{A1}, z_{B1}, z_{S1}} [z_{A1}^{(1-\gamma)\lambda} z_{S1}^{\gamma\lambda}]$$

subject to

$$w_1 = p_{A1}z_{A1} + (p_{B1} - p_{C1})z_{B1} + p_{C1}z_{S1}$$

$$z_{S1} \geq z_{B1},\ (z_{A1}, z_{B1}) \geq 0$$

We can now solve the problem using the Lagrangean method. Let $\mu_{11} \geq 0$ and $\mu_{12} \geq 0$ be the multipliers corresponding to the two constraints. The Lagrangean function of the problem is

$$L_1 = z_{A1}^{(1-\gamma)\lambda} z_{S1}^{\gamma\lambda} + \mu_{11}[w_1 - p_{A1}z_{A1} - (p_{B1} - p_{C1})z_{B1} - p_{C1}z_{S1}]$$
$$+ \mu_{12}(z_{S1} - z_{B1})$$

The first order (Kuhn-Tucker) conditions of the problem are

$$\frac{\partial L_1}{\partial z_{A1}} = (1-\gamma)\lambda\ z_{A1}^{(1-\gamma)\lambda-1} z_{S1}^{\gamma\lambda} - \mu_{11}p_{A1} \leq 0 \qquad \frac{\partial L_1}{\partial z_{A1}} z_{A1} = 0$$

$$\frac{\partial L_1}{\partial z_{B1}} = -\mu_{11}(p_{B1} - p_{C1}) - \mu_{12} \leq 0 \qquad \frac{\partial L_1}{\partial z_{B1}} z_{B1} = 0$$

$$\frac{\partial L_1}{\partial z_{S1}} = \gamma\lambda z_{A1}^{(1-\gamma)\lambda} z_{S1}^{\gamma\lambda-1} - \mu_{11}p_{C1} = 0$$

$$\frac{\partial L_1}{\partial \mu_{11}} = w_1 - p_{A1}z_{A1} - (p_{B1} - p_{C1})z_{B1} - p_{C1}z_{S1} = 0$$

$$\frac{\partial L_1}{\partial \mu_{12}} = z_{S1} - z_{B1} \geq 0 \qquad \frac{\partial L_1}{\partial \mu_{12}} \mu_{12} = 0$$

From the first and third conditions follows that z_{A1} and z_{S1} respectively are positive. In the opposite case, i.e., if one of them or both are zero, then the first term of these conditions (the marginal utility) will go to infinity, violating the non-negativity in the first condition and the equality in the third condition. Assuming now that $p_B > p_C$, i.e., the price of a new good is higher than the price of a used good, and given that μ_{11} is positive (from the fact that z_{A1} and z_{S1} are positive), it is implied by the second condition that the

optimal quantity of new goods that the consumer will buy is zero, i.e., the second condition will hold with strict inequality.

The optimal solution is

$$z^*_{A1} = \frac{(1-\gamma)w_1}{p_{A1}}, \quad z^*_{S1} = \frac{\gamma w_1}{p_{C1}} \quad \text{and} \quad z^*_{B1} = 0$$

From $z_{S1} = z_{B1} + z_{C1} + v_{C1}$, we have

$$z^*_{C1} = \frac{\gamma w_1}{p_{C1}} - v_{C1}$$

which can be positive or negative. If z_{C1} is positive the consumer buys used goods, while if it is negative the consumer sells part of his used goods inventory.

Substitution of the optimal values back in the utility function will yield the maximum expected utility which is

$$f_1(w_1;p_1,v_{C1}) = A(p_1)w_1$$

where

$$A(p_1) = [(1-\gamma)^{(1-\gamma)}\gamma^{\gamma}/p_{A1}^{(1-\gamma)}p_{C1}^{\gamma}]^{\lambda}.$$

The optimal solution of the one-period problem showed that as long as the price of the new goods is higher than the price of the used goods and the probability of rationing is zero, the consumer will buy (or sell) only used goods in the last period of the horizon. Of course, if the price of new goods is lower than the price of the used goods then the consumer's optimal reaction will be to try to sell all his used goods inventory and buy cheaper new goods, thus achieving a higher utility. This second case appears to be of no empirical significance, therefore I will not examine it formally.

The Two-Period Horizon Problem

Suppose now that the consumer's horizon consists of two periods. The first period is called period 2 (i.e., two periods remaining in the horizon) and the second (last) period is called period 1. In the beginning of the horizon the consumer's wealth is assumed to be

$$w_2 = \text{Bonds} + p_{C2}v_{C2}$$

i.e., the bonds that he bought in the past and can cash now, plus his inventory of used goods evaluated in current (period 2) prices.

After the consumer decides upon the consumption allocation for period 2, as explained earlier, the wealth becomes

$$w_2 = p_{A2}z_{A2} + (p_{B2} - p_{C2})z_{B2} + p_{C2}z_{S2} + T_2 \qquad (3.1)$$

where T_2 represents the value of one-period bonds bought during period 2. Let now H_2 represent the savings (portfolio) of the consumer in period 2. We have

$$H_2 = w_2 - p_{A2}z_{A2} - p_{C2}z_{S2} \qquad (3.2)$$

or, by using (3.1),

$$H_2 = T_2 + (p_{B2} - p_{C2})z_{B2} \qquad (3.3)$$

Equation (3.3) shows that portfolio consists of bonds, T_2, and the amount invested in durable goods, $(p_{B2} - p_{C2})z_{B2}$.

In the beginning of the next period (i.e., period 1), the consumer will cash in his bonds, receiving a return r per dollar, and will have his inventory of durable goods, that now will classify as used, evaluated at period's 1 prices, i.e.

$$w_1 = rT_2 + p_{C1}z_{B2}$$

or by using (3.3) to substitute for T_2,

$$w_1 = rH_2 + [p_{C1} - r(p_{B2} - p_{C2})]z_{B2}. \qquad (3.4)$$

The first term in the right hand side (RHS) of (3.4) gives the "certain" portfolio return, i.e., the return if all savings were invested in the riskless bond. The second term shows the difference in returns made by investing in the durable good. If the second term is positive then the investment in durable goods was more profitable than the investment in bonds and vice versa. Note that $z_{B2} = v_{C1}$, i.e., the new goods bought and used in period 2 will consist in the inventory of used goods in period 1.

Define π_{B2} as the portfolio's share invested in the risky durable good, i.e.,

$$\pi_{B2} = \frac{(p_{B2} - p_{C2})z_{B2}}{H_2} \tag{3.5}$$

with $0 \leq \pi_{B2} \leq 1$.

Using (3.5), we can now rewrite (3.4) as

$$w_1 = H_2[r + (\frac{p_{C1}}{p_{B2} - p_{C2}} - r)\pi_{B2}]$$

or by setting $\beta_1 = p_{C1}/(p_{B2} - p_{C2})$

$$w_1 = H_2[r + (\beta_1 - r)\pi_{B2}] \tag{3.6}$$

The functional form of the value function of the two period horizon problem is

$$f_2(w_2;p_2,v_{C2}) = \max_{z_{A1},z_{B2},z_{S2}} \{z_{A2}^{(1-\gamma)\lambda} z_{S2}^{\gamma\lambda} + \alpha \mathop{E}_{p_1} f_1(w_1;p_1,z_{B2})\}$$

By using expressions (3.5) and (3.6) to substitute for w_1 and z_{B2} respectively we get

$$\begin{aligned} f_2(w_2;p_2,v_{C2}) = \max_{z_{A2},z_{S2},\pi_{B2}} &\{z_{A2}^{(1-\gamma)\lambda} z_{S2}^{\gamma\lambda} \\ &+ \alpha \mathop{E}_{p_1} f_1(H_2[r + (\beta_1-r)\pi_{B2}]; \\ &p_1, \frac{H_2\pi_{B2}}{(p_{B2} - p_{C2})}\} \end{aligned} \tag{3.7}$$

subject to

$$w_2 \geq p_{A2}z_{A2} + p_{C2}z_{S2} + H_2\pi_{B2} \tag{3.8}$$

$$z_{S2} \geq \frac{H_2\pi_{B2}}{p_{B2} - p_{C2}}, \quad (z_{A2},\pi_{B2}) \geq 0,$$

where p_1, p_2 are the price vectors of the corresponding periods. The first constraint is the wealth constraint minus the investment in the riskless asset. The second constraint says that the consumption of services is greater or equal to the amount of new goods bought by the consumer. In equation (3.7), I transformed the consumer's problem into one involving a consumption choice for period 2 (i.e., choosing optimal values for z_{A2} and z_{S2}) and an investment choice (i.e., choosing an optimal value for the share invested in durable goods). Using now the form that gives the maximum expected utility for period one which I derived in the end of the previous section and observing that the first term in the RHS of (3.7) is independent of π_{B2}, I can rewrite (3.7) as

$$f_2(w_2;p_2,v_{C2}) = \max_{z_{A2},z_{S2}} \{z_{A2}^{(1-\gamma)\lambda} z_{S2}^{\gamma\lambda} + \alpha \max_{\pi_{B2}} \mathop{E}_{p_1} A(p_1)H_2^{\lambda}[r + (\beta_1 - r)\pi_{B2}]^{\lambda}\}\} \tag{3.9}$$

subject to (3.8).

The second term in the RHS of (3.9) constitutes the portfolio problem of the consumer and can be examined independently. Let the expected portfolio returns be represented by the function

$$G_1(\pi_{B2},p_{B2} - p_{C2}) = \mathop{E}_{p_1} A(p_1)[r + (\beta_1 - r)\pi_{B2}]^{\lambda}. \tag{3.10}$$

This function is concave in π_{B2} and given that π_{B2} takes values in the compact set (0,1), it has a unique maximum, for each $(p_{B2} - p_{C2})$. Next,

I examine some properties of the portfolio share π_{B2} that maximizes (3.10). Let $G_{1\pi}$ be the derivative of (3.10) with respect to π_{B2}, i.e.

$$G_{1\pi}(\pi_{B2},p_{B2} - p_{C2}) = \underset{p_1}{E}\ A(p_1)\lambda[r + (\beta_1-r)\pi_{B2}]^{\lambda-1}(\beta_1-r). \quad (3.11)$$

Next, examine this derivative as π_{B2} takes its extreme values. For $\pi_{B2} = 0$, (3.11) becomes

$$G_{1\pi}(0,p_{B2} - p_{C2}) = \underset{p_1}{E}\ A(p_1)\lambda r^{\lambda-1}(\beta_1-r).$$

If $p_{B2} - p_{C2}$ is large, i.e., the price of the new goods is considerably larger than the price of the used good so that the cost per unit of investment in durable goods is large, then the investment return on durable β_1 will be small and consequently $G_{1\pi}(0,p_{B2} - p_{C2})$ will be negative. Thus, a low expected return β_1 makes the consumer reluctant to invest in durable goods, so that $\pi^*_{B2} = 0$. From the other side, if the difference $p_{B2} - p_{C2}$ is small, the return β_1 will be high and $G_{1\pi}(0,p_{B2} - p_{C2})$ will be positive leading to a positive value for the optimal share of the investment in durable goods. In other words, it will be profitable for the consumer to invest part of his portfolio in the new durable good. For $\pi_{B2} = 1$, (3.11) becomes

$$G_{1\pi}(1,p_{B2} - p_{C2}) = \underset{p_1}{E}\ A(p_1)\lambda\beta_1^{\lambda-1}(\beta_1-r).$$

When $(p_{B2} - p_{C2})$ is sufficiently small, β_1 will be large and $G_{1\pi}(1,p_{B2} - p_{C2})$ will be positive. This implies that the consumer's optimal portfolio strategy will be to invest all his savings in the durable good, i.e., $\pi^*_{B2} = 1$.

The above analysis suggests that there exist values $(p_{B2} - p_{C2})^m$ and $(p_{B2} - p_{C2})^u$ defined by

$$G_{1\pi}(1,(p_{B2} - p_{C2})^m) = 0 \quad \text{and} \quad G_{1\pi}(0,(p_{B2} - p_{C2})^u) = 0$$

such that, if $p_{B2} - p_{C2}$ takes values inside the open interval $((p_{B2} - p_{C2})^m, (p_{B2} - p_{C2})^u)$, then the optimal share of durable goods in the consumer's portfolio will be given by

$$G_{1\pi}(\pi^*_{B2}, p_{B2} - p_{C2}) = 0 \tag{3.11'}$$

with $0 < \pi^*_{B2} < 1$.

As it was shown above, if $p_{B2} - p_{C2}$ is greater or equal to $(p_{B2} - p_{C2})^u$ then $\pi^*_{B2} = 0$, while if $p_{B2} - p_{C2}$ is smaller or equal to $(p_{B2} - p_{C2})^m$ then $\pi^*_{B2} = 1$.

Next, let $k_1(p_{B2} - p_{C2})$ be the maximum expected portfolio return, i.e.

$$k_1(p_{B2} - p_{C2}) = \underset{p_1}{E} \; A(p_1)[r + (\beta_1 - r)\pi^*_{B2}]^\lambda \tag{3.12}$$

with $0 \le \pi^*_{B2} \le 1$. Using (3.12), I can rewrite (3.9) and (3.8) as

$$f_2(w_2; p_2, c_{C2}) = \max_{z_{A2}, z_{S2}} \{z_{A2}^{(1-\gamma)\lambda} z_{S2}^{\gamma\lambda} + \alpha \, H_2^\lambda k_1(p_{B2} - p_{C2})\} \tag{3.13}$$

subject to

$$H_2(1 - \pi^*_{B2}) \ge 0, \; z_{S2} \ge \frac{H_2 \pi^*_{B2}}{p_{B2} - p_{C2}}, \; z_{A2} \ge 0$$

The first constraint is derived by substituting expression (3.2), which gives H_2, into the wealth constraint given in (3.8). Let now $\mu_{21} \ge 0$ and $\mu_{22} \ge 0$ be the multipliers for the two constraints in (3.13) and form the Lagrangean

$$L_2 = z_{A2}^{(1-\gamma)\lambda} z_{S2}^{\gamma\lambda} + \alpha \, H_2^\lambda k_1 + \mu_{21} H_2(1 - \pi^*_{B2}) + \mu_{22}\left(z_{S2} - \frac{H_2 \pi^*_{B2}}{p_{B2} - p_{C2}}\right)$$

The first order (Kuhn-Tucker) conditions of the problem are

$$\frac{\partial L_2}{\partial z_{A2}} = (1-\gamma)\lambda z_{A2}^{(1-\gamma)\lambda - 1} z_{S2}^{\gamma\lambda} - \alpha\lambda k_1 H_2^{\lambda-1} p_{A2} - \mu_{21} p_{A2}(1 - \pi^*_{B2}) + \mu_{22} \frac{p_{A2}\pi^*_{B2}}{p_{B2} - p_{C2}} = 0 \tag{3.14}$$

$$\frac{\partial L_2}{\partial z_{S2}} = \gamma\lambda z_{A2}^{(1-\gamma)\lambda} z_{S2}^{\gamma\lambda-1} - \alpha\lambda k_1 H_2^{\lambda-1} p_{C2} - \mu_{21} p_{C2}(1 - \pi^*_{B2}) +$$

$$\mu_{22} \frac{p_{C2}\pi^*_{B2}}{p_{B2} - p_{C2}} = 0$$

$$\frac{\partial L_2}{\partial \mu_{21}} = H_2(1 - \pi^*_{B2}) \geq 0 \qquad \frac{\partial L_2}{\partial \mu_{21}} \mu_{21} = 0$$

$$\frac{\partial L_2}{\partial \mu_{22}} = z_{S2} - \frac{H_2 \pi^*_{B2}}{p_{B2} - p_{C2}} \geq 0 \qquad \frac{\partial L_2}{\partial \mu_{22}} \mu_{22} = 0$$

The first two conditions hold with equality because both z_{A2} and z_{S2} are positive. If they were zero, the respective marginal utilities, i.e., the first terms of the two conditions, will be infinity which will violate the conditions. Also the value of the portfolio (savings) H_2 is positive since otherwise the second term of the two conditions will be infinity. The third condition $\frac{\partial L_2}{\partial \mu_{21}} \mu_{21} = 0$ can be written as $H_2(1 - \pi^*_{B2})\mu_{21} = 0$. From the fact that H_2 is positive it follows that $(1 - \pi^*_{B2})\mu_{21} = 0$. Thus, the third term of the first two conditions is equal to zero.

Examine now the last condition. Depending upon the price difference $(p_{B2} - p_{C2})$ the constraint may or may not be effective. Define now the critical value $(p_{B2} - p_{C2})^k$ such that

$$(p_{B2} - p_{C2})^k = \frac{H_2 \pi^*_{B2}((p_{B2} - p_{C2})^k)}{z_{S2}}.$$

This suggests that if $p_{B2} - p_{C2} > (p_{B2} - p_{C2})^k$ the fourth condition holds with strict inequality, which in turn implies that $\mu_{22} = 0$ and the constraint is ineffective, while if $p_{B2} - p_{C2} \leq (p_{B2} - p_{C2})^k$ then the constraint is effective. Note that when the constraint is effective,

the consumer consumes only services coming from new durable goods, i.e., the consumer sells all his inventory of used goods and buys new ones.

Consider now the case where the fourth condition holds with inequality, i.e., when $\mu_{22} = 0$, or $p_{B2} - p_{C2} > (p_{B2} - p_{C2})^k$. Solution of the first order conditions yield the following demand functions in this region

$$z^*_{A2} = \frac{(1-\gamma)w_2}{p_{A2}} \Omega_1 \tag{3.15}$$

$$z^*_{S2} = \frac{\gamma w_2}{p_{C2}} \Omega_1 \tag{3.16}$$

where

$$\Omega_1 = \frac{(\alpha k_1)^{\frac{1}{\lambda-1}}}{(\alpha k_1)^{\frac{1}{\lambda-1}} + A(p_2)^{\frac{1}{\lambda-1}}}$$

and $$A(p_2) = \left[\frac{(1-\gamma)^{(1-\gamma)}\gamma^{\gamma}}{p_{A2}^{(1-\gamma)}\, p_{C2}^{\gamma}}\right]^{\lambda}$$

The demand function for new durable goods z^*_{B2} can be derived from the optimal portfolio share of the durable goods $\pi^*_{B2}(p_{B2} - p_{C2})$ (which is calculated from equation (3.12)) and the definition of the portfolio share (equation (3.5)). Thus:

$$z^*_{B2} = \frac{H^*_2 \pi^*_{B2}(p_{B2} - p_{C2})}{p_{B2} - p_{C2}} \tag{3.17}$$

where: $H^*_2 = w_2 - p_{A2}z^*_{A2} - p_{C2}z^*_{S2}$.

Finally from the relation

$$z^*_{C2} = z^*_{S2} - z^*_{B2} - v_{C2} \tag{3.18}$$

we calculate the excess demand function for used durable goods.

Next consider the case where $z_{S2} = \dfrac{H_2\pi_{B2}}{p_{B2} - p_{C2}}$ or $\mu_{22} \geq 0$, i.e., $p_{B2} - p_{C2} < (p_{B3} - p_{C2})^k$. In other words, the constraint is effective. Solution of the first order conditions (3.14) in this region yields

$$z^*_{A2} = \frac{(1-\gamma)w_2}{p_{A2}} \Omega_2 \tag{3.19}$$

$$z^*_{S2} = \frac{w_2}{p_{C2}} \Omega_2 \tag{3.20}$$

$$z^*_{B2} = z^*_{S2} \tag{3.21}$$

$$z^*_{S2} = -v_{C2} \tag{3.22}$$

where

$$\Omega_2 = \frac{p_{C2}\pi^*_{B2}}{\gamma(p_{B2} - p_{C2}) + p_{C2}\pi^*_{B2}}.$$

As we can see, functions (3.19) and (3.20) are almost similar to (3.15) and (3.16), except for the Ω terms. The fact that Ω's are functions of the prices p_{A2}, p_{B2}, p_{C2} makes the study of the effects of prices upon the demand functions non-trivial. In the next section, I examine the properties of the above derived demand functions with respect to prices.

Properties of the Demand Functions

In the first part of this section I derive the properties of the demand functions (3.15-3.18), which consist the most important case. In the last part I discuss the signs of the demand function (3.19)-(3.22).

Differentiation of the demand function (3.15) with respect to p_{A2}, yields

$$\frac{\partial z^*_{A2}}{\partial p_{A2}} = -\frac{(1-\gamma)w_2}{p_{A2}{}^2} \Omega_1 + \frac{(1-\gamma)w_2}{p_{A2}} \frac{\partial \Omega_1}{\partial p_{A2}} \tag{3.23}$$

But

$$\frac{\partial \Omega_1}{\partial p_{A2}} = \frac{\frac{\lambda(1-\gamma)}{(\lambda-1)p_{A2}} (\alpha k_1)^{\frac{1}{\lambda-1}} A(p_2)^{\frac{1}{\lambda-1}}}{[(\alpha k_1)^{\frac{1}{\lambda-1}} + A(p_2)^{\frac{1}{\lambda-1}}]^2} < 0 \tag{3.24}$$

because $\frac{\partial k_1}{\partial p_{A2}} = 0$, $\frac{\partial A(p_2)}{\partial p_{A2}} = -\frac{\lambda(1-\gamma)A(p_2)}{p_{A2}}$ and $\lambda < 1$.

From (3.23) and (3.24) it is evident that

$$\frac{\partial z^*_{A2}}{\partial p_{A2}} \quad 0 \tag{3.25}$$

From the first order conditions (3.14), the following relation can be derived

$$z_{S2} = \frac{\gamma p_{A2} z_{A2}}{(1-\gamma)p_{C2}} \tag{3.26}$$

Differentiation (3.26) with respect to p_{A2} yields

$$\frac{\partial z^*_{S2}}{\partial p_{A2}} = \frac{\gamma}{(1-\gamma)p_{C2}} [z_{A2} + p_{A2} \frac{\partial z_{A2}}{\partial p_{A2}}]$$

The term in the brackets, using (3.23) and (3.15), can be written as

$$z_{A2} + p_{A2} \frac{\partial z_{A2}}{\partial p_{A2}} = (1-\gamma)w_2 \frac{\partial \Omega_1}{\partial p_{A2}}$$

Therefore, I obtain

$$\frac{\partial z^*_{S2}}{\partial p_{A2}} = \frac{\gamma w_2}{p_{C2}} \frac{\partial \Omega_1}{\partial p_{A2}} < 0 \tag{3.27}$$

The negative sign is derived by using (3.24).

From the demand for new goods (3.17), differentiating with respect to p_{A2}, I derive

$$\frac{\partial z^*_{B2}}{\partial p_{A2}} = \frac{\pi^*_{B2}}{p_{B2} - p_{B2}} \frac{\partial H_2}{\partial p_{A2}}$$

But

$$\frac{\partial H_2}{\partial p_{A2}} = -(z_{A2} + p_{A2}\frac{\partial z_{A2}}{\partial p_{A2}} + p_{C2}\frac{\partial z_{S2}}{\partial p_{A2}}) = -w_2\frac{\partial \Omega_1}{\partial p_{A2}}$$

by using (3.23), (3.15) and (3.27). Therefore

$$\frac{\partial z^*_{B2}}{\partial p_{A2}} = -\frac{w_2 \pi^*_{B2}}{p_{B2} - p_{C2}}\frac{\partial \Omega_1}{\partial p_{A2}} > 0. \tag{3.28}$$

And from (3.18), using (3.27) and (3.28),

$$\frac{\partial z^*_{C2}}{\partial p_{A2}} = \frac{\partial z^*_{S2}}{\partial p_{A2}} - \frac{\partial z^*_{B2}}{\partial p_{A2}} < 0 \tag{3.29}$$

The interpretation of the signs derived in (3.25), (3.27), (3.28), (3.29), lies in the fact that as the price of the perishable good A goes up, its quantity demanded goes down and also current consumption becomes less favorable compared with next period's consumption due to the higher price level. Thus, the consumer will cut back the current consumption of both good A and services S in favor of higher savings. The increase in savings will increase the demand for new durable goods, since their optimal portfolio share π^*_{B2} is unaffected by changes in p_{A2}. The increase in z_{B2} is compatible with a decrease in z_{S2} by a larger decrease in z_{C2} as (3.29) described.

Let us examine now the effects of a change in the prices of the new and used durable goods, p_{B2} and p_{C2}, upon the demand functions (3.15-3.18). The complication that arises now is that changes in p_{B2} and p_{C2} will affect the portfolio shares. I start by examining the later effect first.

Differentiation of (3.11') with resepct to p_{B2} yields

$$G_{1\pi\pi}(\pi^*_{B2}, p_{B2} - p_{C2})\frac{d\pi^*_{B2}}{dp_{B2}} + G_{1\pi p_B}(\pi^*_{B2}, p_{B2} - p_{C2}) = 0$$

Therefore

$$\frac{d\pi^*_{B2}}{dp_{B2}} = - \frac{G_{1\pi p_B}}{G_{1\pi\pi}} \tag{3.30}$$

Since $G_{1\pi\pi}$ is negative, due to the concavity of G_1, the sign of (3.30) depends upon the sign of the cross derivative $G_{1\pi p_B}$. Examine now that sign. From (11), differentiating with respect to p_{B2}, I obtain

$$G_{1\pi p_B} = \underset{p_1}{E}\; A(p_1)\lambda\{[r + (\beta_1 - r)\pi_{B2}]^{\lambda-1}(- \frac{\beta_1}{p_{B2} - p_{C2}}) +$$

$$(\beta_1 - r)(\lambda - 1)[r + (\beta_1 - r)\pi_{B2}]^{\lambda-2}(- \frac{\beta_1 \pi_{B2}}{p_{B2} - p_{C2}})\}$$

or:

$$G_{1\pi p_B} = \underset{p_1}{E}\; A(p_1)\lambda[r + (\beta_1 - r)\pi_{B2}]^{\lambda-2}(- \frac{\beta_1}{p_{B2} - p_{C2}})[r + \lambda(\beta_1 - r)\pi_{B2}]$$

The term $r + (\beta_1 - r)\pi_{B2}$ shows the value of one dollar investment at the beginning of the next period. This term is positive, except in the case where the consumer invests all his savings in new durable goods and at the beginning of next period the price of the used goods falls to zero. It is easy to see that $r + \lambda(\beta_1 - r)\pi_{B2}$ is positive. If $\beta_1 \geq r$ then the positive sign follows, while if $\beta_1 < r$ then we can see that $r + \lambda(\beta_1 - r)\pi_{B2}$ is strictly greater than $r + (B_1 - r)\pi_{B2}$ because $\lambda < 1$, and given that $r + \lambda(\beta_1 - r)\pi_{B2}$ is non-negative, the positive sign follows. This proves that $G_{1\pi p_B}$ is negative. Hence

$$\frac{\partial \pi^*_{B2}(p_{B2} - p_{C2})}{\partial p_{B2}} < 0, \tag{3.31}$$

and by using the same method of proof

$$\frac{\partial \pi^*_{B2}(p_{B2} - p_{C2})}{\partial p_{C2}} > 0. \tag{3.32}$$

In other words, as the price of the new durable good goes up, the consumer will reduce the share of durable goods in his portfolio, increasing thus the share of the riskless asset. Also, if the used goods price (or services' price) increases, then the price of the investment in durable goods $(p_{B2} - p_{C2})$ will fall and the consumer will react by increasing the portfolio share of durable goods.

Differentiating now the maximum portfolio return function (3.12), I obtain

$$\frac{\partial k_1(p_{B2} - p_{C2})}{\partial p_{B2}} = G_{1\pi} \frac{d\pi^*_{B2}}{dp_{B2}} + G_{1p_{B2}}.$$

The first term in the RHS is always equal to zero from the optimality of π^*_{B2}, while we can show that the second term is negative. Indeed, differentiating (3.10) with respect to p_{B2} gives

$$G_{1p_B} = \underset{p_1}{E}\, A(p_1)\lambda[r + (\beta_1 - r)\pi_{B2}]^{\lambda-1} \left(- \frac{\beta_1}{p_{B2} - p_{B2}}\right) < 0.$$

Therefore

$$\frac{\partial k_1(p_{B2} - p_{C2})}{\partial p_{B2}} < 0 \tag{3.33}$$

and, by the same method,

$$\frac{\partial k_1(p_{B2} - p_{C2})}{\partial p_{C2}} > 0. \tag{3.34}$$

Next, I examine the effects of changes in the services price p_{C2} upon the demand functions (3.15)-(3.18). Differentiating (3.15) with respect to p_{C2} gives

$$\frac{\partial z^*_{A2}}{\partial p_{C2}} = \frac{(1-\gamma)w}{p_{A2}} \frac{\partial \Omega_1}{\partial p_{C2}} \tag{3.35}$$

But

$$\frac{\partial \Omega_1}{\partial p} = \frac{\frac{1}{\lambda-1}(\alpha k_1)^{\lambda-1}\ A(p_2)^{\lambda-1}}{[(\alpha k_1)^{\frac{1}{\lambda-1}} + A(p_2)^{\frac{1}{\lambda-1}}]^2} \cdot [(\alpha k_1)^{-1} \frac{\partial k_1}{\partial p_{C2}} - A(p_2)^{-1} \frac{\partial A(p_2)}{\partial p_{C2}}]$$

Given that

$$\frac{\partial A(p_2)}{\partial p_{C2}} = - \frac{\lambda\gamma}{p_{C2}} A(p_2) < 0,$$

it follows that

$$\frac{\partial \Omega_1}{\partial p_{C2}} < 0 \qquad (3.36)$$

and from (3.35),

$$\frac{\partial z^*_{A2}}{\partial p_{C2}} > 0 \qquad (3.37)$$

Differentiating now (3.26) with respect ot p_{C2} and using (3.37) gives

$$\frac{\partial z^*_{S2}}{\partial p_{C2}} = \frac{\gamma p_{A2}}{(1-\gamma)p_{C2}} [- \frac{z_{A2}}{p_{C2}} + \frac{\partial z_{A2}}{\partial p_{C2}}] < 0 \qquad (3.38)$$

Differentiating (3.17) with respect to p_{C2}, I obtain

$$\frac{\partial z^*_{B2}}{\partial p_{C2}} = \frac{(\pi^*_{B2} \frac{\partial H^*_2}{\partial p_{C2}} + H^*_2 \frac{\partial \pi^*_{B2}}{\partial p_{C2}})(p_{B2} - p_{C2}) + H^*_2\pi^*_{B2}}{(p_{B2} - p_{C2})^2} > 0. \qquad (3.39)$$

The positive sign follows from (3.32) and from

$$\frac{\partial H_2}{\partial p_{C2}} = - \frac{p_{A2}}{1-\gamma} \frac{\partial z_{A2}}{\partial p_{C2}} > 0.$$

Finally, differentiation of (3.18), and use of (3.38) and (3.39) yields

$$\frac{\partial z^*_{C2}}{\partial p_{C2}} = \frac{\partial z^*_{S2}}{\partial p_{C2}} - \frac{\partial z^*_{B2}}{\partial p_{C2}} < 0. \qquad (3.40)$$

The above derived signs follow from the fact that an increase in the price of the durable good services p_{C2} will make current consumption less desirable against future consumption, causing a decrease in the quantity consumed of both goods A and S, and an increase in savings. The increase in savings, combined with the increase in the expected returns of the inventment in new durable goods$\left(\frac{p_{C1}}{p_{B2} - p_{C2}}\right)$, will increase the demand for new durable goods z_{B2}. The increase in z^*_{B2} and the decrease in z^*_{S2} are achieved through a decrease in the quantity demanded of used durable goods z_{C2}, as shown by (3.40).

Next I examine the effects of a change in the price of the new durable goods p_{B2} upon the demand functions (3.15)-(3.18). Differentiating function (3.15) with respect to p_{B2}, I derive

$$\frac{\partial z^*_{A2}}{\partial p_{B2}} = \frac{(1-\gamma)w_1}{p_{A2}} \frac{\partial \Omega_1}{\partial p_{B2}} \quad 0 \tag{3.41}$$

because

$$\frac{\partial \Omega_1}{\partial p} = \frac{\frac{\alpha}{\lambda-1}(\alpha k_1)^{\lambda-1} \quad A(p_2)^{\lambda-1}}{[(\alpha k_1)^{\frac{1}{\lambda-1}} + A(p_2)^{\frac{1}{\lambda-1}}]^2} \frac{\partial k_1}{\partial p_{B2}} > 0$$

by using (3.33)

From differentiation of (3.26) follows:

$$\frac{\partial z^*_{S2}}{\partial p_{B2}} = \frac{\gamma p_{A2}}{(1-\gamma)p_{C2}} \frac{\partial z^*_{A2}}{\partial p_{B2}} > 0 \tag{3.42}$$

From differentiation of (3.17) follows:

$$\frac{\partial z^*_{B2}}{\partial p_{B2}} = \frac{\left(\pi^*_{B2} \frac{\partial H^*_2}{\partial p_{B2}} + H^*_2 \frac{\partial \pi^*_{B2}}{\partial p_{B2}}\right)(p_{B2} - p_{C2}) - H^*_2 \pi^*_{B2}}{(p_{B2} - p_{C2})^2} < 0 \tag{3.43}$$

because of (3.31) and of

$$\frac{\partial H^*_2}{\partial p_{B2}} = -p_{A2} \frac{\partial z^*_{A2}}{\partial p_{B2}} - p_{C2} \frac{\partial z^*_{S2}}{\partial p_{B2}} < 0.$$

Finally from (3.18), (3.42) and (3.43), it follows that:

$$\frac{\partial z^*_{C2}}{\partial p_{B2}} = \frac{\partial z^*_{S2}}{\partial p_{B2}} - \frac{\partial z^*_{B2}}{\partial p_{B2}} > 0. \tag{3.44}$$

Thus, an increase in the price of a new durable good makes current consumption of both goods A and S more desirable, reducing desired savings. It also increases the price of the investment in new durable goods ($p_{B2} - p_{C2}$), reducing, therefore, its expected return. The reduction in savings, combined with the decrease in the expected return of the investment in new durable goods, decreases the quantity demanded of the new durable good and increases the demand for used durable goods.

Let us now recapitulate the derived results. According to the above analysis, the demand functions have the following signs:

$$\begin{aligned} z^*_{A2} &= z_{A2}(\overset{(-)}{pA2}, \overset{(+)}{p_{B2}}, \overset{(-)}{p_{C2}}) \\ z^*_{B2} &= z_{B2}(\overset{(+)}{p_{A2}}, \overset{(-)}{p_{B2}}, \overset{(+)}{p_{C2}}) \\ z^*_{C2} &= z_{C2}(\overset{(-)}{p_{A2}}, \overset{(+)}{p_{B2}}, \overset{(-)}{p_{C2}}) \end{aligned} \tag{3.45}$$

Using the same method as above we can derive the signs for the demand functions (3.19-3.22). Such an analysis gives the following signs:

$$\begin{aligned} z^*_{A2} &= z_{A2}(\overset{(-)}{p_{A2}}, \overset{(-)}{p_{B2}}, \overset{(+)}{p_{C2}}) \\ z^*_{B2} &= z_{B2}(\overset{(0)}{p_{A2}}, \overset{(-)}{p_{B2}}, \overset{(?)}{p_{C2}}) = z^*_{S2} \\ z^*_{C2} &= -v_{C2} \end{aligned} \tag{3.46}$$

Remember here that demand functions (3.46) are derived under the assumption that $(p_{B2} - p_{C2}) \leq (p_{B2} - p_{C2})^k$ which means that the investment in new durable goods is so favorable that the consumer will sell all his

used goods and consume services from new goods. The reversal of some of the signs and the indeterminacy of others stems from the fact that changes in the prices of new and used goods will work amongst others factprs to relax or strengthen the effect of the constraint. For example, if p_{B2} increases and/or p_{C2} decreases, then the constraint is partially relaxed, because now the investment in durable goods becomes less attractive, and this will lead to an decrease in the attractiveness of savings against current consumption. Therefore, we can expect a decrease in the current consumption of goods A and S. Note here that the effect of a change in p_{C2} upon the consumption of durable goods services is indeterminate. The reason is that an increase in p_{C2} will cause two conflicting effects. First, as we saw above, it will relax the constraint increasing the attractiveness of savings and decreasing the demand for z_{S2}, and second, it will make the consumption of S more attractive than the consumption of A. The relative strength of the two effects will determine the direction of the sign. (It can be shown that, if $\gamma > \pi_{B2}$ then $\frac{\partial z^*_{B2}}{\partial p_{C2}}$ is positive but not the opposite.)

Concluding Remarks

After the demand functions and their properties were derived for the representative consumer, the derivation of the market demand functions is in order. The aggregation process poses a series of questions to the researcher. For example, how do we aggregate over persons with different utility functions, wealth levels, inventory levels, expectation formation functions, etc.? These questions remain unanswered. For the purposes of the present work, I will follow the traditional approach, i.e., I will ignore the problems, and I will assume that the market

demand for new durable goods and the excess demand for used durable goods are derived by simple summation of the individuals' functions. Also,I will assume that the prices of the new and used durable goods are in the region to which the functions in (3.45) hold. In other words, I will use the functions in (3.45) as the market demand functions.

The derived demand and excess demand functions present a major improvement over the previously used functions. Out of the analysis of the representative consumer, assuming uncertainty about future prices, I manage to derive demands as functions of only the current prices. In the existing literature, as we saw in the introduction, the demands are functions of future prices, which, of course, are assumed known.

One of the major assumptions of the analysis is the full execution of all the orders the consumers place. A more realistic assumption is that the consumer faces the probability of being rationed in the market. This assumption makes the role of used goods inventories much more significant and adds one more dimension in the problem, i.e., the consumer invests in durable goods not only as a portfolio return maximizing asset values but also as an insurance against rationing the next period. In Appendix B, I analyze the problem with positive probabilities of rationing. As the interested reader can see, the complexity of the problem increases and the derivation of exact forms for the demand functions becomes very difficult. Consequently, the study of the properties of these demand functions is even more difficult. All that I will show in the Appendix is that the demand functions will depend upon the same variables as in (3.45). I expect that the properties of the functions in (3.45) will carry over to the more general case where the probabilities of rationing are positive.

Finally, a note about the length of the horizon and the durability of the product. It is not difficult to extend the horizon to any number of periods. A larger horizon is not expected to change any of the derived results. Dropping the two period durability assumption in favor of an n-period durability of the product will increase the number of the state variables of the problem. With a good that lasts n-periods, we will have n-1 kinds of used goods in the market each period and this implies that the consumers face n prices and may keep up to n-1 different kinds of used durable goods inventories. Such an assumption is not expected to alter the derived results, but explicit derivation of demand functions and their properties will be unmanageable.

CHAPTER FOUR

OPTIMAL BEHAVIOR OF THE SUPPLIERS IN THE MARKETS FOR NEW AND USED DURABLE GOODS: THE NON-COOPERATIVE CASE

Introduction

This chapter examines the behavior of the suppliers in the markets for new and used durable goods and the trading properties that result from this behavior. The supplier of new goods is assumed to be a monopolist. The existence of a price-setter is assumed for the SHM. This last assumption can be justified by two scenarios. One scenario is to assume that in the SHM there is only one dealer who controls the market. The other scenario is to assume that there are many small SHM dealers who collectively set the price, possibly through an agent. The current situation in the used automobile market supports this second scenario. The role of the price setting agent is played by the publisher of the so-called "Blue Book" which serves as an indicator of the used cars prices. The price of the "Blue Book" is the fee the agent receives for his price setting services. In the following analysis the word "dealer" is used to signify the price setter under both scenarios. It is evident that actions of the new goods monopolist will affect the behavior of the used goods dealer and vice versa. In what follows I will proceed by first examining the behavior of each participant separately, assuming that he knows the behavior of the other participant in the market. Later we will remove the above separation and will examine the behavior of both participants simultaneously.

In the next section I introduce the assumptions about the structure of the market system, which consists of the two markets for new and used goods, and the assumptions about the behavior of the monopolist and the SHM dealer. The third section will examine the behavior of the monopolist of new durable goods, assuming that he knows the distribution of the price of the used goods. In the fourth section I examine the derivation of the SHM dealer's optimal behavior, assuming that he knows the optimal behavior of the monopolist derived in the third section. The fifth section reexamines the behavior of both the monopolist and SHM dealer and derives the Nash equilibrium of the system, i.e., the equilibrium behavior that results by assuming that each participant behaves under the assumption that the other will follow his optimal strategy. Also in this section I examine the price and quantity probability distribution functions. The final section gives some concluding remarks.

The Structure of the Durable Goods Market System

As the analysis of the previous chapter showed, the monopolist and the SHM dealer will face a demand and an excess demand, respectively, for their goods as functions of both the prices of new and used goods. To simplify the analysis, I will ignore the prices of the other goods and given that our analysis examines a partial equilibrium, I will ignore the wealth variable. Furthermore, I will assume that the two functions are linear with the respective signs as derived in (3.45), and that both functions are stationary up to an additive stochastic term. The stochastic term can be thought as taking into account changes in the distribution of wealth or differences in entrance and exit of consumers in the market during the horizon. Changing Chapter Three's terminology

to conform to the existing literature I will asume that the demand for new durable goods has the form

$$D = g_1(r,p) + u_1 \tag{4.1}$$

where

$$g_1(r,p) = a_0 + a_1 r + a_2 p \tag{4.2}$$

and

$$(a_0, a_1, -a_2) > 0,\ a_1 > -a_2.$$

The price of the new goods is p and has a negative effect upon the demand for new goods, while the price of used goods (which in the following analysis will be called the "rental rate") is r and has a positive effect upon the demand for new goods. Furthermore, the own price effect is assumed stronger than the rental rate effect.

The excess demand for used goods has the form

$$d = g_2(r,p) + u_2 \tag{4.3}$$

where

$$g_2(r,p) = b_0 + b_1 r + b_2 p$$

with $\quad (b_0, -b_1, b_2) > 0,\ -b_1 > b_2.$

Here the price of the new goods p has a positive effect and the rental rate has a negative effect. The rental rate effect is assumed dominant here.

The stochastic terms u_1 and u_2 follow the joint distribution function $\Phi_{12}(u_1,u_2)$ supported on $[-L_1,U_1 \times -L_2,U_2]$ with $(L_1,L_2,U_1,U_2) > 0$. Distribution function $\Phi_1(u)$ is the marginal distribution for u_1 supported on $[-L_1,U_1]$ and $\Phi_2(u)$ is the marginal distribution for u_2 supported on $[-L_2,U_2]$. The expected values for both stochastic terms are zero, while their standard deviations are σ_1(for u_1) and σ_2(for u_2) and their covariance is σ_{12}.

The new durable goods monopolist is assumed to set his optimal price and quantity supplied each period before the realization of his demand (4.1) in order to maximize the present value of his expected profits over an infinite horizon. Price and quantity setting will result in excess supply or demand. The monopolist keeps any excess supply as inventory and backlogs any excess demand. The backlogging can be treated as negative inventory.

The monopolist's information set is assumed to contain the inventory (positive or negative) carried from the previous period,the expected demand (4.2), the distribution functions and some information about the behavior of the SHM dealer. Specifically, in the next section I proceed under the assumption that the monopolist knows the current rental rate r and the distribution function of the future rental rates. In Section E, reconsidering the monopolist's problem, I assume that he knows the optimal strategy of the SHM dealer as derived in Section D.

The monopolist produces the new durable good under a linear production cost function where the cost of production level q is given by

$$C = c_1 q, \quad c_1 > 0 \tag{4.5}$$

The existence of the inventories causes the monopolist to incur inventory holding and backlogging costs. The inventory holding costs are mainly storage and maintenance costs incurred when the inventory is positive. The backlogging costs may include goodwill losses, management of delayed orders and, assuming that all the orders are paid in full as they are received by the monopolist, it may include interest and penalty payments for the undelivered orders. I assume that both costs can be

combined in an inventory holding-backlogging cost function whose form is quadratic, i.e.,

$$h = h_0 + h_1 y + \frac{h_2}{2} y^2, \quad (h_0, h_2) > 0, \tag{4.6}$$

where y is the (positive or negative) inventory and the sign of h will depend upon the relative importance of the storage versus backlogging costs. When storage is costlier than backlogging, h_1 is positive and vice versa. Note here that I assume that new durable goods kept as inventory do not depreciate. For example, furniture or refrigerators are sold as new even if many months have passed from their production date. The same is true for cars. A 1986 model which is produced during August 1985 can be sold as a new car during September 1986, i.e., it can remain as inventory for more than a year.

The assumed form of the demand for new goods (4.1) allows, at high prices, for the awkward possibility of negative demand. Following the literature I will introduce a price upper limit that will exclude the possibility of negative demand. Thus, assume $D \geq 0$ or from (4.1)

$$0 \leq p \leq \bar{p}(r) \tag{4.7}$$

where $\bar{p}(r)$ is determined by $g_1(r, \bar{p}(r) = -L_1$. Hence

$$\bar{p}(r) = \frac{L_1 - (a_0 + a_1 r)}{a_2}. \tag{4.8}$$

In order to ensure that the monopolist will find it worthwhile to produce, I will introduce a profitability condition. This condition requires that the expected marginal revenue at the price upper limit (i.e., at output L_1) will be above the marginal cost of producing this output. As you can see this condition is a restatement in expectation terms of the certainty case condition that requires the marginal cost to

be below the price at zero output. Futhermore, the profitability condition in the uncertainty case is only a necessary condition but not a sufficient one. If, for example, the inventory cost is very high, then the profitability condition is not sufficient to induce positive positive production. Using (4.1) and (4.5) we can calculate the condition $EMR(q = L_1) > MC(q = L_1)$ or:

$$a_2x_1 + a_0 + q_1r > 2L_1. \tag{4.9}$$

Condition (4.9) presents a problem. Given that the monopolist has no control over r, changes in the rental rate may reverse the inequality. For example, if r is very low, then according to our demand structure of the problem most of the consumers will shift to the SHM, so that the demand for new goods will become very small and (4.9) may not hold. To avoid further complications of an already complex problem (the complexity will become more and more evident as we proceed), I assume that (4.9) holds for every r. A more plausible assumption would be that the condition holds for an upper segment of the rental rate interval, but this complicates the analysis disproportionately.

Let us now examine the behavior of the SHM dealer. The dealer's function is to transfer used goods among consumers. The dealer makes his profit by the spread between buying and selling price and, possibly, capital gains on his stock of the used good. Here again, for simplicity I assume that this spread is zero. For the determination of an optimal spread see Zabel (26). Given that the dealer does not produce goods, he is only going to incur inventory holding-backlogging costs. Using the

same reasoning as for the monopolist's case, I assume that the dealer faces the following inventory holding-backlogging function:

$$L = l_0 + l_1 x + \frac{l_2}{2} x^2, \; (l_0, l_2) > 0 \tag{4.10}$$

where x is the used good inventory held by the dealer and l_1's sign depends upon the relative cost of storage and backlogging.

The dealer sets the rental rate in order to maximize the present value of his expected profits over an infinite horizon, facing the stochastic demand function (4.3) and the cost function (4.10). The dealer's information set includes his current (positive or negative) inventory, the demand distributions and the monopolist's optimal strategy, i.e., reaction functions.

The excess demand function (4.3) takes positive or negative values. The excess demand is positive if the buyers demand a larger quantity than the quantity the sellers want to sell. In this case the dealer tries to cover the excess demand with his inventory and backlogs any remaining excess demand. The excess demand function takes negative values when the sellers want to sell larger quantities than the buyers demand. In this case the dealer adds the difference to his inventory. But there is a limit as to how negative the excess demand can be. The sellers cannot sell more goods than the quantity they possess and this is the quantity of new goods delivered to them the previous period. Let T be the quantity of used goods in the consumers' hands at the beginning of this period. Also, I will assume that the rental rate will never be larger than the price of the new goods. In other words, I introduce the constraint:

$$0 \leq r \leq p. \tag{4.11}$$

When the rental rate is equal to the price of the new goods then it is expected that almost all the consumers who own used goods will sell them and will try to buy new goods. But if all the consumers try to replace their used units with new ones then there is a high probability of backlogging. This in return implies that some consumers will remain unsatisfied. I will assume that some consumers want to consume the services of the durable good so badly that they will keep some units of the used good to ensure themselves against the probability of rationing even in the case where the rental rate is the same as the price of the new good. This assumption translates to:

$$g_2(p) - L_2 > -T \tag{4.12}$$

for any p. Condition (4.12) simplifies greatly the analysis by reducing the number of constraints, without altering the results in any crucial way.

Price and Supply Reactions in the Market for New Durable Goods

In this section I proceed under the assumption that the monopolist's information set includes the inventory level y, the demand function (4.1) and its properties, the production and inventory holding-backlogging cost functions (4.5) and (4.6), the current rental rate r set by the SHM dealer and the distribution of the future rental rates $\Phi_3(\tilde{r})$. Given this information the monopolist sets his price p and supply s (where $s = q + y$, i.e., the supply consists of the current production and the inventory), in order to maximize the present value of the expected profits over an infinite horizon. The maximization takes place over two constraints. The first is the price constraint given in (4.7) and the other is a quantity constraint that ensures the

non-negativity of the production level, i.e., $s \geq y$. The monopolist's value function satisfies the equation

$$F(y,r) = \max_{s,p}\{pg_1(r,p) - c_1(s-y) - \int h(s-g_1(r,p)-u_1)d\Phi_1(u) +$$

$$\alpha\int\int F(s-g_1(r,p)-u_1,r)d\Phi_1(u)d\Phi_3(\tilde{r})\} \qquad (4.13)$$

subject to

$$0 \leq p \leq \bar{p}(r), \; s \geq y. \qquad (4.14)$$

The first term in the RHS (right hand side) of (4.13) is the expected revenue, the second is the production cost, the third is the expected inventory holding-backlogging cost and the last term is the present value of the maximum expected profit of all the remaining periods assuming that the optimal strategy will be followed during these periods (Bellman's Principle of Optimality). This last term is discounted by a factor α taking values in the inverval (0,1).

Before I proceed to determine the optimal value for the choice variables s and p, the existence, uniqueness and concavity of the infinite horizon value function that satisfies (4.13) has to be established. The most common method for the above proof is to show that the single-period horizon value function exists which is unique and concave and subsequently, to use arguments advanced by Bellman (1) or Denardo (8) to pass to the limit of a finite horizon model. Assume that only one period is remaining in the horizon and that the valuation of the inventory remaining in the end of the horizon given by $f_0(y)$ is assumed without loss of generality to be zero (it can be any concave function). The single period horizon function satisfies the equation

$$F_1(y,r) = \max_{s,p}\{pg_{1(r,p)} - c_1(s-y) - \int h(s-g_1(r,p)-u_1)d\Phi_1(u)\} \qquad (4.15)$$

Let $G_1(s,p,y,r)$ be the maximand of (4.15). Given r, which in the present case is a given parameter, inspection reveals that G_1 is concave in its three remaining arguments. Using Lemma 1 by Iglehart (10), transformed to apply for concave functions, it can be shown that $F_1(y,r)$ is concave in y. Continuation of the same line of arguments will prove that for any n-period horizon, $F_n(y,r)$ is concave in y. Now using the functional form of $F(y,r)$ given in (4.13) we can easily show that the contraction and monotonicity assumptions (see Appendix B) required for Denardo's (8) theorems hold. Therefore, there exists a unique function $F(y,r)$ that satisfies the infinite horizon functional equation. Also, F is the limit towards which F_n converges uniformly. The concavity of F_n with respect to y and its uniform convergence ensure that $F(y,r)$ is concave in y.

Now we can proceed to the solution of the infinite horizon problem. Let $\lambda \geq 0$ and $\delta \geq 0$ be the multipliers corresponding to the constraints $s \geq y$ and $\bar{p}(r) \geq p$. Also, let $G(s,p.y.r)$ be the maximand of $F(y,r)$. The Kuhn-Tucker conditions of the problem described in (4.13) subject to (4.14) are

$$\begin{aligned}
&\text{(a)}\quad D_sG(s^*,p^*,y,r) + \lambda = 0 && && (4.16)\\
&\text{(b)}\quad D_pG(s^*,p^*,y,r) - \delta \leq 0 && [D_pG(s^*,p^*,y,r) - \delta]p^* = 0\\
&\text{(c)}\quad \bar{p}(r) - p^* \geq 0 && [\bar{p}(r) - p^*]\delta = 0\\
&\text{(d)}\quad s^* - y \geq 0 && (s^* - y)\lambda = 0
\end{aligned}$$

where

$$D_sG(s,p,y,r) = -c_1 - [\int h'(s-g_1-u_1)d\Phi_1(u) - \alpha\int\int F_y(s-g_1-u_1,r)d\Phi_1(u)d\Phi_3(\tilde{r})] \qquad (4.17)$$

$$D_pF(s,p.y,r) = g_1 + pg_{1p} + g_{1p}[\int h'(s-g_1-u_1)d\Phi_1(u) - \alpha\int\int F_y(s-g_1-u_1,r)d\Phi_1(u)d\Phi_3(\tilde{r})] \qquad (4.18)$$

where F_y is the derivative of F with respect to its first argument and g_{1p} is the derivative of $g_1(r,p)$ with respect to p. To simplify the exposition I suppress the arguments of the various functions except in the cases where they are introduced for first time or assume particular values. Note that substitution of (4.17) and (4.18) yields

$$D_pG = g_1 + g_{1p}(p-c_1-D_sG). \tag{4.19}$$

In what follows, I analyze conditions (4.16) to determine the optimal solution. The solution will vary according to whether the constraints are effective or not.

Consider first the case when the price constraint is effective, i.e., $p^* = \bar{p}(r)$. This implies that condition (4.16)(b) is satisfied with equality, i.e., $D_pG = \delta$. Substituting in (4.19) and using (4.16)(a), I derive

$$D_pG(s^*,\bar{p}(r),y,r) = g_1(r,\bar{p}(r)) + g_{1p}(r,\bar{p}(r))[\bar{p}(r)-c_1-\lambda] = \delta$$

or by using (4.8) to substitute for $\bar{p}(r)$

$$2L_1 - (a_2c_1 + a_0 + a_1r) = \delta - a_2\lambda \geq 0$$

The non-negativity results from the non-negativity of the two multipliers and the negativity of a_2. The above result contradicts the profitability condition (4.9) which requires that

$$2L_1 - (a_2c_1 + a_0 + {}_1r) < 0.$$

This contradiction implies that the monopolist will never charge a price as high as $\bar{p}(r)$. In other words, the price constraint $p \leq \bar{p}(r)$ is always ineffective and therefore the multiplier δ is always equal to zero.

The above analysis suggests that the problem has only one constraint that may be effective, the quantity constraint. Consider next the case

where the quantity constraint is ineffective, i.e., $s > y$. In this case, condition (4.16)(d) requires $\lambda = 0$. The optimal solution now will satisfy the Kuhn-Tucker conditions with equality, i.e., it will be the interior solution

$$D_sG(s^*,p^*,y,r) = 0 \tag{4.20}$$
$$D_pG(s^*,p^*,y,r) = 0$$

Indeed, given that both multipliers are zero, the only case for an extreme solution is when $p^* = 0$. But when $p^* = 0$ then (4.19) becomes:

$$D_pG(s^*,0,y,r) = g_1(r,0) - a_2c_1 > 0$$

which contradicts the condition (4.16)(b) given that $\delta = 0$. Thus, the optimal price will be positive and given by the interior solution (4.20).

Examine now the case where the quantity constraint is effective, i.e., $s = y$. If the inventory is very high then intuition suggests that the monopolist to economize on storage costs, will set a zero price to reduce this inventory. Let us examine if this is true. Suppose that $p^* = 0$. Then equation (4.18) becomes

$$D_pG(s,0,s,r) = g_1 + g_{1p}[\int h'd\Phi_1(u) - \alpha\int\int F_y d\Phi_1(u)d\Phi_3(\tilde{r})]. \tag{4.21}$$

To determine the sign of the above equation we determine the following limits that hold for any p

$$\lim_{s\to\infty} \int h'(s-g_1-u_1)d\Phi_1(u) = \infty \tag{4.22}$$

$$\lim_{s\to-\infty} \int h'(s-g_1-u_1)d\Phi_1(u) = -\infty$$

due to the convexity of the function h.

Also, from the single-period horizon functional equation given in (4.15) after substitution of the optimal solution $s^*(y,r)$, $p^*(y,r)$ and

differentiation with respect to y, we obtain

$$F_{1y}(y,r) = c_1 + D_pG_1(s^*,p^*,y,r) \frac{\partial p^*(y,r)}{\partial y} +$$

$$D_sG_1(s^*,p^*,y,r) \frac{\partial s^*(y,r)}{\partial y}$$

We can now use the K-T conditions (4.16), which apply to the single period problem too, to examine the limits of the above equation. As we show above, the price constraint (4.16)(c) is never effective. Therefore, condition (4.16)(b) suggests that the second term in the RHS of the above equation will always be zero. The sign of the third term in the RHS can be determined by conditions (4.16)(a) and (4.16)(d). In particular for y sufficiently large, the quantity constraint will be effective, i.e., $\lambda \geq 0$ from (4.16)(d). In this case (4.16)(a) implies that the sign of the third term is negative or zero. If now y is sufficiently small so that (4.16)(d) is satisfied with inequality then $\lambda = 0$ and (4.16)(a) implies that the third term is zero. In technical terms, the above discussion suggests that

$$F_{1y}(y,r) \leq c_1 \quad , \quad \lim_{y\to-\infty} F_{1y}(y,r) = c_1$$

It is easy to show that for the two-period horizon problem

$$F_{2y}(y,r) \leq c_1 \quad , \quad \lim_{y\to-\infty} F_{2y}(y,r) = c_1$$

Thus, by induction and a limiting argument we can show that

$$F_y(y,r) \leq c_1 \quad , \quad \lim_{y\to-\infty} F_y(y,r) = c_1 \tag{4.23}$$

Using the limits in (4.22) and (4.23) it follows that (4.21) has the following limits

$$\lim_{s\to\infty} D_pG(s,0,s,r) = -\infty$$

$$\lim_{s\to-\infty} D_pG(s,0,s,r) = \infty$$

This result suggests the existence of an inventory level $\hat{y}(=\hat{s})$ such that

$$D_pG(\hat{s},0,\hat{y},r) = 0.$$

It is easy now to see that for any actual inventory larger or equal to $\hat{y}$ the optimal price will be zero, while for inventories smaller than $\hat{y}$ the optimal price will be positive, i.e., for $y < \hat{y}$ the optimal price is given by

$$D_pG(y,p^*(y,r),y,r) = 0 \qquad (4.24)$$

with $p^*(y,r) > 0$ and $s = y$.

The next question now is at what level of inventory (supply) the quantity constraint will cease being effective? To answer this question, ignore the constraint for a moment. Taking the limits of equation (4.17) and using (4.22) and (4.23) we can show that, for any p, it holds that

$$\lim_{s\to\infty} D_sG(s,p,y,r) = -\infty$$

$$\lim_{s\to-\infty} D_sG(s,p,y,r) = \infty$$

The above limits, holding for any p, suggest that there exists a supply level $s^*(r)$ such that

$$D_sG(s^*(r),p^*(s,r),y,r) = 0.$$

It is now evident that if the actual inventory is smaller than $s^*(r)$ then the production will be positive, i.e., the quantity constraint will be ineffective. In the case where the inventory is larger or equal to $s^*(r)$, then the production stops, i.e., the quantity constraint becomes effective.

The above analysis suggests that if the actual inventory is larger or equal to $s^*(r)$ but smaller that $\hat{y}$, then the optimal price will be

positive. Let us examine the price behavior in this inventory (supply) interval. Total differentiation of condition (4.24) yields:

$$\frac{dp^*(y,r)}{dy} = -\frac{D_{py}G}{D_{pp}G} \tag{4.25}$$

Further differentiation of derivative (4.18) with respect to p and s(=y) results in

$$D_{pp}G(s,p,y,r) = 2g_{1p} - g_{1p}^2[\int h''d\Phi_1(u) - \alpha\int\int F_{yy}d\Phi_1(u)d\Phi_3(\tilde{r})] < 0$$

$$D_{ps}G(s,p,y,r) = g_{1p}[\int h''d\Phi_1(u) - \alpha\int\int F_{yy}d\Phi_1(u)d\Phi_3(\tilde{r})] < 0$$

where, F_{yy} is the second derivative of F with respect to its first argument. The signs are due to the convexity of the inventory cost function and the concavity of the value function with respect to y. The above derived signs indicate that expression (4.25) is negative, which in turn implies that in the interval $[s^*(r),\hat{y}]$ as the actual inventory y declines towards $s^*(r)$, the optimal price $p^*(y,r)$ increases.

Recapitulating the results I derived up to now, we see that the optimal solution has three phases depending upon the level of inventory. These are:

Phase I, when the inventory is larger than $\hat{y}$, then the optimal solution is $p^* = 0$, $s^* = y$ (or $q^* = 0$).

Phase II, when the inventory is in the interval $[s^*(r),\hat{y}]$, then the optimal solution is $p^*(y,r)$ calculated form (4.24) and $s^* = y$ (or $q^* = 0$).

Phase III, when the inventory is lower than $s^*(r)$, then the optimal solution is given by the interior solution calculated from (4.20).

Let us now consider the calculation of the interior solution. The exact calculation of the optimal price and supply requires knowledge of

the general form of the value function. As will be shown below, the general form that satisfies the infinite horizon functional equation (4.13), when the optimal solution for the current and all future periods is interior, is

$$F(y,r) = \Theta_1 y + \Theta_2 r + \frac{\pi_2}{2} r^2 + \rho_1 \tag{4.26}$$

In other words, the branch of the value function corresponding to Phase III has the form given above, if it can be shown that when the inventory enters the interval corresponding to that Phase, it stays there forever.

Using (4.26), (4.2), (4.5) and (4.6) we can rewrite the derivatives in (4.20) as

$$\begin{aligned} D_s G(s^*,p^*,y,r) &= 0c_1 + \alpha\Theta_1 - h_2[s^*-g_1(r,p^*)] = 0 \\ D_p G(s^*,p^*,y,r) &+ g_1(r,p^*) + a_2 p^* - a_2(\alpha\Theta_1-h_1) + \\ &a_2 h_2[s^*-g_1(r,p^*)] = 0 \end{aligned} \tag{4.27}$$

Solution of the above system with respect to p^* and s^* yields

$$p^*(r) = \frac{a_2 c_1 - a_0 - a_1 r}{2a_2} \tag{4.28}$$

$$s^*(r) = \frac{2(\alpha\Theta_1 - h_1 - c_1) + h_2(a_0 + a_1 r + a_2 c_1)}{2h_2} \tag{4.29}$$

As expected, the optimal solution is independent of the level of inventory y. This independence stems from the linearity of the production cost function. As is shown by Zabel (27), a more general production cost function will result in an optimal solution that will depend upon the inventory level. In what follows we should always keep in mind this limitation of the present model. Solution (4.28)-(4.29) describes the monopolist's reaction functions with respect to the pricing decisions of the SHM dealer. Taking the derivatives of (4.28)

and (4.29) we can easily show that the relation between both p^* and s^* with r is positive. In other words, the higher the rental rate the SHM dealer charges, the more customers will shift to the new goods market. Thus, the monopolist will find it profitable to increase both his price and supply.

Next, I proceed to derive conditions under which the inventory will enter and remain forever in the Phase III interval, i.e., the solution will be interior forever, after a possible initial transitory period. Suppose initially that the actual inventory is higher than $s^*(r)$. As we saw previously, in this case the monopolist will stop production, $q^* = 0$ and will charge a price lower than $p^*(r)$. Thus, the inventory will eventually be depleted and enter the Phase III interval. The Phase III inventory interval does not have a lower limit, because, however small the inventory, the monopolist will produce a quantity such that the supply will always be $s^*(r)$.

Next, suppose that the inventory is already smaller than $s^*(r)$ and examine the possibility of the inventory becoming larger. The maximum expected inventory for next period y_{emax} is achieved if current demand is realized at its lower limit, i.e.,

$$y_{emax} = s^*(r) - g_1(r,p^*(r)) + L_1$$

However, since $D \geq 0$, it follows that $g_1(r,p^*(r)) - L_1 \geq 0$ and

$$s^*(r) \geq y_{emax}.$$

This suggests that once the inventory enters its the Phase III interval it stays there forever.

The above analysis showed that once the inventory enters the interior solution interval, it stays there forever, without any additional conditions required. We can now proceed to calculate the

parameters of the general form (4.26) using the "unknown coefficients" determination method.

Substitution of (4.26), the optimal solution (4.28)-(4.29) and the explicit forms of the demand and cost functions into the functional form (4.13) gives

$$F(y,r) = c_1 y - \frac{a_1(a_0+a_2c_1)}{2a_2} r - \frac{a_1^2}{4a_2} r^2 + D \tag{4.30}$$

where

$$D = \frac{(\alpha\Theta_1-h_1-c_1)^2}{2h_2} - \frac{(a_0+a_2c_1)^2}{4a_2} + \alpha\rho_1 - h_0 - \frac{h_2}{2}\sigma_1^2 + \alpha\Theta_2\bar{r} +$$

$$\frac{\alpha\pi_2}{2}(\bar{r}^2 + \sigma_r^2)$$

with $\bar{r}$ and σ_r^2 as the expected value and variance of the rental rate. It is apparent that the above derived form of the value function is consistent with the general form assumed in (4.26). Equating the corresponding coefficients for y,r and r^2 I derive

$$\Theta_1 = c_1 > 0 \tag{4.31}$$

$$\Theta_2 = - \frac{a_1(a_0+a_2c_1)}{2a_2}$$

$$\pi_2 = - \frac{a_1^2}{2a_2} > 0.$$

The value of σ_1 is not important and its computation is omitted. As it was to be expected, the value function is convex in r.

Following Zabel (27), we can define target or equilibrium levels for the inventory and the choice variables. Given that we have not yet derived the final solutions for the choice variables, we postpone the calculation of their equilibrium values for Section E. In what follows, I will calculate the equilibrium value for the inventory.

Define as equilibrium inventory, the level at which next period's expected initial inventory is equal to current period's initial inventory. In this problem equilibrium inventory is easy to calculate since, given r, next period's expected initial inventory is a constant amount depending on the constant price $p^*(r)$ and constant supply $s^*(r)$. From the definition we have

$$y_e = s^*(r) - g_1(r,p^*(r)) = \frac{\alpha\Theta_1 - h_1 - c_1}{h_2}$$

and given that $\Theta_1 = c_1$,

$$y_e = \frac{(\alpha - a)c_1 - h_1}{h_2} \tag{4.32}$$

As we can see the equilibrium inventory is independent of the rental rate r. The sign of the equilibrium inventory depends upon the sign of h_1. If the storage is more or equally expensive as backlogging, that is h_1 is non-negative, then the equilibrium inventory will be negative. If backlogging is more costly, i.e., h_1 is negative, then the sign of y_e depends upon the sign of the numerator in (4.32).

Optimal Policy in the Second Hand Market

In this section I analyze the SHM dealer's decision-making process. I assume now that the dealer's information set, apart from the demand and inventory cost functions, contains the monopolist's optimal reaction functions. The independence of the monopolist's reaction functions from the new goods inventory enables us to transform the SHM into one with one state variable, the used goods inventory x, and one choice variable, the rental rate r. Indeed, using the reaction function (4.28) we can

rewrite the expected excess demand for used goods as

$$g_2(r) = b_0 + b_2 \frac{a_2c_1-a_0}{2a_2} + (b_1 - \frac{a_1b_2}{2a_2})\ r$$

$$= d_0 + d_1r,\ (d_0,-d_1) > 0. \tag{4.33}$$

Also substitution of (4.28) in the rental rate constraint (4.11) gives

$$0 \le r \le r^0 \quad , \quad r^0 = \frac{a_2c_1-a_0}{2a_2+a_1} \tag{4.34}$$

Given now the excess demand function, the inventory holding-backlogging function (4.10), the inventory x and the marginal distribution function $\Phi_2(u)$, the dealer sets the rental rate r in order to maximize the present value of his expected profits over an infinite horizon, under the constraint (4.34). The dealer's value function satisfies the equation

$$f(x) = \max_r\{rg_2(r) - \int l(x-g_2(r)-u_2)d\Phi_2(u) +$$

$$\alpha\int f(x-g_2(r)-u_2)d\Phi_2(u)\} \tag{4.35}$$

subject to (4.34).

The first term in the RHS of (4.35) gives the expected revenue from buying and selling used goods, the second term is the expected inventory cost, and the last term is the discounted maximum expected profits for all the remaining periods in the horizon, assuming that the optimal policy will be followed during that period.

The existence, uniqueness and concavity of the infinite horizon value function for the SHM can be established in the same way as in the monopolist's problem. In particular, starting from the single period horizon problem we can show that the one-period horizon value function will be concave due to the linearity of the excess demand function and

the convexity of the inventory holding-backlogging functions. Extending the horizon to n periods and using the contraction-mapping and monotonicity arguments advanced by Denardo (8), we can show that the infinite horizon value function exists, is unique and is concave in x (see also note 1).

Let now $J(r,x)$ be the maximand of (4.35) and $D_rJ(r,x)$ be the derivative with respect to r. The first order conditions of the problem (4.35) subject to (4.34), given that $\mu \geq 0$ is the multiplier, are

$$D_rJ(r^*,x) - \mu \leq 0 \qquad [D_rJ(r,x) - \mu]r^* = 0 \tag{4.36}$$
$$(r^0 - r^*) \geq 0 \qquad (r^0 - r^*)\mu = 0$$

We can now show that if the used goods inventory is very high, then the dealer will set the rental rate at zero in order to deplete his inventory quickly and save on inventory holding costs, while if the backlog is very large, he will charge a price equal to the rental rate's upper limit in order to satisfy part of his backlog. In other words, we can show that:

$$\lim_{x\to\infty} D_rJ(r,x) = -\infty \tag{4.37}$$
$$\lim_{x\to-\infty} D_rJ(r,x) = \infty$$

Equations (4.37) hold because

$$\lim_{x\to\infty} \int l'(x-g_2-u_2)d\Phi_2(u) = \infty \tag{4.38}$$
$$\lim_{x\to-\infty} \int l'(x-g_2-u_2)d\Phi_2(u) = -\infty$$

and

$$\lim_{x\to\infty} \int f'(x-g_2-u_2)d\Phi_2(u) = -\infty \tag{4.39}$$
$$\lim_{x\to-\infty} \int f'(x-g_2-u_2)d\Phi_2(u) = \infty$$

The limits in (4.39) are derived by starting from the single period horizon functional equation and using (4.38) to prove that (4.39) holds for that horizon. Subsequently, using induction and a limiting argument we can show that (4.39) holds for the infinite horizon value function.

The limits in (4.37) given that they hold for any r, and the concavity of f(x) suggest that there are two inventory levels x_u and x_m such that if the actual inventory is above x_u the dealer charges a zero rental rate, while if the actual inventory is below x_m the dealer charges the rental rate upper limit r^0. Next I proceed to calculate these inventory limiting values.

Consider first the case where $r^* = r^0$ and $\mu = 0$. From (4.36) we can see that x_m is given by

$$D_r J(r^0, x_m) = 0 \tag{4.40}$$

If the actual inventory is lower than x_m, then $r^* = r^0$ and $\mu > 0$ or $D_r J(r^*, x) > 0$.

Consider next the case where $r^* = 0$ and calculate x_u from

$$D_r J(0, x_u) = 0. \tag{4.41}$$

If the actual inventory lies above x_u, then $D_r J(0, x_u)$ will be negative and the optimal price will still be zero.

Let us examine now the relation between rental rate and inventory assuming that the inventory lies in the interval (x_m, x_u). Further differentiation of $D_r J(x,r)$ with respect to r and x, yields:

$$D_{rr} J(r,x) = 2g_{2r} - g_{2r}^2[\int l''(x-g_2-u_2)d\Phi_2(u) - \alpha\int f''(x-g_2-u_2d\Phi_2(u)] < 0$$

$$D_{rx} J(r,x) = g_{2r}[\int l''(x-g_2-u_2)d\Phi_2(u) - \alpha\int f''(x-g_2-u_2)d\Phi_2(u)] < 0$$

Given that in the interval (x_m, x_u) the first order condition (4.36) is satisfied with equality, it holds that:

$$\frac{dr^*(x)}{dx} = - \frac{D_{rx}J(r,x)}{D_{rr}J(r,x)} < 0,$$

in other words, as x increases the rental rate r decreases. Therefore, when $x_m < x < x_u$, the optimal rental rate satisfies $r^0 > r^* > 0$ and is calculated from

$$D_r J(r^*,x) = 0. \tag{4.42}$$

As it will be shown below, assuming that the current and all future solutions are interior, i.e., x lies always in the interval (x_m, x_u), the general form of the value function f(x) is

$$f(x) = \Theta_3 x + \frac{\pi_f}{2} x^2 + \rho_2 \tag{4.43}$$

Using (4.43), (4.10) and (4.33), I can write the derivative of the maximand J(r,x) as

$$D_r J(r,x) = d_0 + 2d_1 r - d_1[(\alpha\Theta_3 - 1) + (\alpha\pi_4 - 1_2)(x - d_0 - d_1 r)] \tag{4.44}$$

Substituting (4.44) into (4.40), (4.41) and (4.42), I can calculate x_m, x_u and $r^*(x)$ respectively

$$x_m = \frac{d_0[1+d_1(\alpha\pi_4 - 1_2)] - d_1(\alpha\Theta_3 - 1_1) + [2+d_1(\alpha\pi_4 - 1_2)]d_1 r^0}{d_1(\alpha\pi_4 - 1_2)} \tag{4.45}$$

$$x_u = \frac{d_0[1+d_1(\alpha\pi_4 - 1_2)] - d_1(\alpha\Theta_3 - 1_1)}{d_1(\alpha\pi_4 - 1_2)} \tag{4.46}$$

$$r^*(x) = \frac{d_1(\alpha\Theta_3\ 1_1) - d_0[1+d_1(\alpha\pi_4 - 1_2)] + d_1(\alpha\pi_4 - 1_2)x}{d_1[2+d_1(\alpha\pi_4 - 1_2)]}. \tag{4.47}$$

The concavity of the value function f(x) implies that the coefficient π_4 is negative. Given the negativity of π_4 and d_1 we can easily see from (4.47) that the level of inventory and the optimal rental rate are inversely related.

Next I proceed to derive conditions that will ensure that the assumption under which the value function has the general form (4.43) holds. In other words, I will derive conditions that guarantee that the inventory enters its interior solution interval and stays there forever. As I will show below, the condition that needs to hold for the inventory to enter its interior solution limits and stay there forever is:

$$d_0 + d_1 r^0 + U_2 < 0 \tag{4.48}$$

i.e., the excess demand upper limit, when the rental rate achieves its maximum value, is negative.

Assume initially that the dealer's inventory is very small, i.e., the backlog is large so that $x < x_m$. In this case the dealer will charge the largest possible rental rate so that the excess demand will be negative, i.e., he will buy used goods from consumers in order to satisfy part of his backlog and save on backlogging costs. In technical terms, for $x < x_m$, then as we saw discussing equation (4.40), μ will be positive and $r^* = r^0$. With $r^* = r^0$, the excess demand will be always negative, provided that condition (4.48) holds, thus backlogging will diminish and eventually x will become larger than x_m.

Assume next that the dealer's inventory is too large, i.e., $x > x_u$. In this case the first order conditions are satisfied by $D_r J(r^*,x) < 0$ which implies that $r^* = 0$. In other words, the dealer gives away his inventory in order to save on storage costs. Given that the excess demand will be positive at zero rental rate, the inventory will be depleted and eventually will fall below x_u. Up to now we showed that the inventory enters its interior solution interval with probability one. Our next concern is to show that it stays in this interval in all future periods.

Suppose initially that the actual inventory during the current period is equal to the lower limit x_m. According to our solution, the optimal rental rate that the dealer will charge is $r^* = r^0$. Let x_{emin} be the minimum expected inventory for next period, i.e.,

$$x_{emin} = x_m - g_2(r^*(x_m)) - U_2 = x_m - (d_0+d_1r^0+U_2)$$

It is obvious that $x_m < x_{emin}$ if (4.48) holds. Even in the worst case where the stochastic term realizes its upper limit, the inventory still remains in its interior solution interval.

Suppose next that $x = x_u$, i.e., the actual inventory is equal to the upper limit of its interior solution interval. The first order conditions imply that $r^*(x_u) = 0$. Therefore, next period's maximum expected inventory x_{emax} will be

$$x_{emax} = x_u - g_2(r^*(x_u)) + L_2 = x_u - (d_0-L_2).$$

Given that $d_0 > L_2$, it holds that $x_u > x_{emax}$. This implies that the inventory never becomes greater than its upper limit once inside the interior solution interval. Thus, I proved that if condition (4.48) holds, the inventory enters its interior solution interval and stays there forever.

Substitution of the optimal solution (4.47) into the value function (4.35) yields:

$$f(x) = \frac{2(\alpha\Theta_3-l_1)-d_0(\alpha\pi_4-l_2)}{2+d_1(\alpha\pi_4-l_2)}\,x \;+\; \frac{\alpha\pi_4-l_2}{2+d_1(\alpha\pi_4-l_2)}\,x^2 \;+\; Q$$

where Q represents the remaining terms which are independent of x.

It is evident that the above equation has the same form as assumed

in (4.43). Using the unknown coefficient methods we derive the following:

$$\Theta_3 = -\frac{2l_1+d_0(\alpha\pi_4-l_2)}{2(1-\alpha)+d_1(\alpha\pi_4-l_2)}$$

$$\pi_4 = \frac{\frac{d_1l_2}{2} - (1-\alpha) + \{[\frac{d_1l_2}{2} - (1-\alpha)]^2 - 2d_1l_2\}^{\frac{1}{2}}}{\alpha d_1} < 0$$

The value of ρ_2 is unimportant. The negative root is chosen for π_4 because of the concavity of the value function. The sign of Θ_3 depends upon the sign of l_1. If backlogging is more expensive than storage, i.e., l_1 is negative, then Θ_3 is negative. Positive l_1 may result in positive or negative Θ_3.

The equilibrium properties of the SHM variables will be discussed in the next section.

A Non-Cooperative (Nash) Equilibrium for the System

In the last two sections, I first derived the monopolist's reaction functions and subsequently the SHM dealer's optimal strategy. This section goes back to the monopolist's problem to derive his optimal strategy under the assumption that the monopolist knows the SHM dealer's optimal policy and expects him to follow it. In other words, I will examine the non-cooperative (Nash) equilibrium of the system. The Nash equilibrium is derived under the assumption that each market participant derives his optimal strategy under the expectation that all other market participants will follow their optimal strategy. After deriving the Nash equilibrium, I will examine the adjustment and equilibrium properties of the system. Given that the purpose here is to study the

stationary properties of the system, I will assume that both inventories of new and used goods are inside their interior solution limits.

The monopolist's value function, under the assumption that he knows the SHM dealer's optimal policy, satisfies the equation

$$F(y,r^*(x)) = \max_{s,p}\{pg_1 - c_1(s-y) - \int h(s-g_1-u_1)d\Phi_1(u) + \alpha\int F(s-g_1-u_1, r^*(x-g_2(r^*)-u_2))d\Phi_{12}(u)\} \tag{4.49}$$

where $g_1 = g_1(r^*(x),p)$ and, as noted earlier, $\Phi_{12}(u)$ is the joint distribution function of $u = (u_1,u_2)$.

To simplify notation, I define

$$N_0 = d_1(\alpha\Theta_3 - 1_1) - d_0[1+d_1(\alpha\pi_4 - 1_2)] \tag{4.50}$$

$$N_1 = d_1(\alpha\pi_4 - 1_2) > 0$$

$$D = d_1[2+d_1(\alpha\pi_4 - {}_2)] < 0$$

Using this notation I can rewrite equation (4.47) as

$$r^*(x) = \frac{N_0 + N_1 x}{D}. \tag{4.51}$$

It follows then that I can write

$$r^*(x-g_2(r^*)-u_2) = \frac{2d_1N_0 - d_0N_1D}{D^2} + \frac{2d_1N_1}{D^2}x - \frac{N_1}{D}u_2 \tag{4.52}$$

$$= M_0 + M_1x + m_2u_2$$

Substituting now (4.52) in (4.49) and using the general form of the value function introduced in (4.25) I obtain

$$F(x,y) = \max_{s,p}\{pg_1 - c_1(s-y) + (\alpha\Theta_1 - h_1)(s-g_1) + \alpha\rho_1 - h_0 + \frac{\alpha\pi_2}{2}[(M_0+M_1x)^2 + M_2^2\sigma_2^2] - \frac{2}{2}[(s-g_1)^2 + \sigma_1^2] + \alpha\Theta_2(M_0+M_1x)\} \tag{4.53}$$

Differentiation of the maximand $G(s,p,y,x)$ of (4.53) with respect to s and p yields:

$$D_s G(s,p,y,x) = -c_1 + (\alpha\Theta_1 - h_1) - h_2[s - a_0 - {}_1 r^*(x) - a_2 p] \qquad (4.54)$$

$$D_p G(s,p,y,x) = a_0 + a_1 r^*(x) + 2a_2 p - a_2[(\alpha\Theta_1 - h_1) - h_2(s - a_0 - a_1 r^*(x) - a_2 p)]$$

where $r^*(x)$ is given in (4.51).

Comparison of conditions (4.54) with the previously used condition (4.27) reveals that the two solutions differ only in that the monopolist now knows the exact form of the marginal distribution of the rental rates, $\Phi_3(\tilde{r})$. Therefore, the monopolist's optimal strategy can be obtained in two ways, either equate (4.54) to zero and solve with respect to s and p, or substitute optimal (4.51) into the reaction functions (4.28) and (4.29). In both cases the optimal solution is:

$$p^*(x) = \frac{(a_2 c_1 - a_0)D - a_1(n_0 + N_1 x)}{2a_2 D} \qquad (4.55)$$

$$s^*(x) = \frac{\alpha\Theta_1 - h_1 - c_1}{h_2} + \frac{(a_2 c_1 + a_0)D + a_1(N_0 + N_1 x)}{2D} \qquad (4.56)$$

These two expressions combined with

$$r^*(x) = \frac{N_0 + N_1 x}{D} \qquad (4.51)$$

give the non-cooperative (Nash) equilibrium of the system. That is, when both agents use the strategies given by (4.51), (4.55) and (4.56), neither can improve his expected profits. As it was noted previously, the linear production cost assumption resulted in an optimal solution independent of the new goods' inventory. A more general production cost function, e.g., quadratic, will entail an optimal solution as functions of both inventories, x and y. But a more general production cost

function will require much stricter conditions to guarantee that the inventories enter their ergodic sets, and thus make the exact calculation of the optimal solution possible.

Examination of the solution (4.51), (4.55), (4.56) easily verifies that all three variables are inversely related to the level of the used goods inventory x. When the used goods inventory is, say, relatively low, the dealer will charge a relatively high rental rate. The high rental rate will shift part of the demand of used goods to the new goods market, enabling the monopolist to charge a higher price and increase his supply. The opposite is true when the used goods inventory is high.

Now I proceed to calculate the equilibrium and adjustment properties of the system. Define first x_e as the used goods equilibrium inventory, i.e., that inventory level at which current inventory equals next period's expected inventory. The equilibrium inventory is given by

$$x_e = x_e - g_2(r^*(x_e))$$

or

$$g_2(r^*(x_e)) = 0$$

Substituting r* given in (4.49) and solving with respect to x_e, I obtain:

$$x_e = -\frac{d_0+d_1(\alpha\Theta_3-l_1)}{d_1(\alpha\pi_4-l_2)}. \tag{4.57}$$

Combination of x_e and the optimal solution (4.51), (4.55), (4.56) makes possible the calculation of the equilibrium values of the choice variables of the system. Letting $r_e = r^*(x_e)$, $p_e = p^*(x_e)$ and

$s_e = s^*(x_e)$, I derive

$$r_e = -\frac{d_0}{d_1} \tag{4.58}$$

$$p_e = \frac{(a_2c_1-a_0)d_1+a_1d_0}{2a_2d_1}$$

$$s_e = \frac{\alpha\Theta_1-h_1-c_1}{h_2} + \frac{(a_2c_1+a_0)d_1-a_1d_0}{2d_1}.$$

Note here that the above equilibrium values coincide with the optimal solution of the static problem, i.e., the case where all the stochastic terms are zero.

Next I provide verification that the values derived in (4.57) and (4.58) are the equilibrium or desired values, by showing that all the variables in the system adjust towards these values.

Substitution of the equilibrium value x_e in next period's expected used goods inventory, yields:

$$x - g_2(r^*) = x_e + \frac{2d_1}{D}(x-x_e). \tag{4.59}$$

That is, next period's expected inventory can be expressed as a function of the equilibrium inventory plus an adjustment factor. Given that the coefficient $\frac{2d_1}{D}$ is positive and smaller than one, when current inventory x is above (below) its equilibrium value x_e then next period's expected inventory will still be above (below) its equilibrium value but it will also be below (above) its current value. In other words, the expected inventory adjusts partially towards its equilibrium value. The size of the adjustment will depend directly upon the coefficient and the difference between current and equilibrium inventories.

For the optimal rental rate the adjustment mechanism is obtained by solving (4.59) with respect to r^* and using the expression given in (4.58) to substitute for r_e

$$r^*(x) = r_e + \frac{N_1}{D}(x - x_e). \tag{4.60}$$

Here the adjustment coefficient is negative and smaller than one, which implies that if the current inventory is higher (lower) than its equilibrium level the SHM dealer will charge a rental rate lower (higher) than the equilibrium rental rate. Furthermore, the closer the current inventory to its equilibrium value, the closer the rental rate to its equilibrium value (and the slower the adjustment).

The corresponding adjustment mechanisms for the monopolist's choice variables are obtained by substituting their equilibrium values derived in (4.58) into their optimal values (4.55) and (4.56) and also using (4.60). They are

$$p^*(x) = p_e - \frac{a_1 N_1}{2a_2 D}(x - x_e) \tag{4.61}$$

$$s^*(x) = s_e + \frac{a_1 N_1}{2D}(x - x_e). \tag{4.62}$$

Both adjustment coefficients are negative, suggesting that both variables adjust to a direction opposite to the inventory x. Furthermore, comparing the adjustment coefficients in (4.60) and (4.61) we can see that the speed of adjustment of the new goods price is lower than the speed of the adjustment of the rental rate, the reason lies in the fact that x affects r more strongly than p.

After showing that all the variables of the system adjust towards their equilibrium values, I proceed to derive the stationary distributions of these variables. As was pointed above, due to the

linearity of the production cost, the inventory of new goods y does not affect the optimal values of the choice variables.

In what follows, I begin by deriving the stationary distribution for the inventory x. Subsequently, I derive the distribution for the choice variables.

Let $\tilde{x}_t$ be the random variable that represents the inventory of the SHM dealer period in t and let x_0 be its initial value at the beginning of the horizon. After the first period trading takes place, the SHM inventory is

$$\tilde{x}_1 = x_0 - g_2(r^*(x_0)) - u_{20} = \frac{2d_1}{D} x_0 + \frac{d_1N_1}{D} x_e - u_{20}$$

After two periods of trading,

$$\tilde{x}_2 = \tilde{x}_1 - g_2(r^*(\tilde{x}_1)) - u_{21} = (\frac{2d_1}{D})^2 x_0 + (\frac{2d_1}{D})(\frac{d_1N_1}{D})(x_e - u_{20})$$
$$+ (\frac{d_1N_1}{D}) x_e - u_{21},$$

and after t periods of trading,

$$\tilde{x}_t = \tilde{x}_{t-1} - g_2(r^*(\tilde{x}_{t-1})) - u_{2t} =$$

$$(\frac{2d_1}{D})^t x_0 + \sum_{j=0}^{t-1} (\frac{2d_1}{D})^{t-(j+1)} (\frac{d_1N_1}{D} x_e - u_{2j}).$$

Now, let $k = t-(j+1)$ so that $\tilde{x}_t$ may also be described by

$$\tilde{x}_t = (\frac{2d_1}{D})^t x_0 + \sum_{k=0}^{t-1} (\frac{2d_1}{D})^k (\frac{d_1N_1}{D} x_e - u_{2t-k-1}).$$

Since u_2 is identically distributed and independent of the parameters of the problem, $\tilde{x}_t$ satisfies

$$\tilde{x}_t = (\frac{2d_1}{D})^t x_0 + \sum_{k=0}^{t-1} (\frac{2d_1}{D})^k (\frac{d_1N_1}{D} x_e - u_{2k}).$$

Finally, since we can write $\frac{d_1N_1}{D} = 1 - \frac{2d_1}{D}$, the random variable $\tilde{x}_t$ can be written as

$$\tilde{x}_t = (\frac{2d_1}{D})^t x_0 + [1-(\frac{2d_1}{D})^t]x_e - \sum_{k=0}^{t-1} (\frac{2d_1}{D})^k u_{2k}$$

Taking the limits as t goes to infinity, we arrive at

$$\tilde{x} = x_e - \sum_{k=0}^{\infty} (\frac{2d_1}{D})^k u_{2k} \qquad (4.63)$$

Since u_{2k} has expected value zero and variance σ_2^2 the above equation implies that

$$E(\tilde{x}) = x_e \qquad (4.64)$$

and

$$\mathrm{Var}\ (\tilde{x}) = \frac{D}{d_1N_1} \sigma_2^2$$

since

$$\frac{1}{1 - (\frac{2d1}{D})^2} = \frac{D}{d_1N_1}$$

The variance of the inventory x is larger than the variance of the used goods demand, since the expression D/d_1N_1 is greater than one. This is the result of the partial adjustment of the choice variables. The following example illustrates that point. Suppose that in the end of period t-1 (or beginning of period t) the used goods inventory is higher than its equilibrium level x_e. From the adjustment mechanisms (4.59) and (4.60) we can see that the dealer will set the rental rate at a level for which the expected inventory at the end of the period t will still be above x_e. If now during period t the demand realizes at its lower limit (i.e., $u_{2t} = -L_2$) the actual inventory at the end of period t will be higher than $x_e + L_2$, i.e., the inventory will fluctuate within

limits wider than the demand limits. As we will see below, if the choice variables adjust totally, as is the new goods market, the variance of the inventory is equal to the variance of the demand.

The stationary distributions for the choice variables are obtained when we substitute the stantionary distribution (4.63) in the adjustment equations (4.60)-(4.62). Defining $\tilde{r} = r^*(\tilde{x})$ and substituting (4.63) into (4.60) we obtain

$$\tilde{r} = r_e - \frac{N_1}{D} \sum_{k=0}^{\infty} \left(\frac{2d_1}{D}\right)^k u_{2k} \tag{4.65}$$

Following the same procedure for p, form (4.61) and (4.63), we obtain

$$\tilde{p} = p_e + \frac{a_1 N_1}{2a_2 D} \sum_{k=0}^{\infty} \left(\frac{2d_1}{D}\right)^k u_{2k} \tag{4.66}$$

Combination of (4.65) and (4.66) enables us to derive properties the joint price distribution of the system

$$\tilde{p} = p_e + \frac{a_1}{2a_2} (r_e - \tilde{r}) \tag{4.67}$$

The following properties are derived from (4.65), (4.66) and (4.67)

$$E(\tilde{r}) = r_e \quad , \quad E(\tilde{p}) = p_e \tag{4.68}$$

$$\mathrm{Var}(\tilde{r}) = \frac{N_1}{d_1 D} S_2^2 \, , \quad \mathrm{Var}(\tilde{p}) = \frac{a_1^2 N_1}{4a_2^2 d_1 D} \sigma_2^2$$

$$\mathrm{Cov}(\tilde{r},\tilde{p}) = E[(\tilde{p}-p_e)(\tilde{r}-r_e)] = - \frac{a_1 N_1}{2a_2 d_1 D} \sigma_2^2$$

Comparison of the two price variances reveals that

$$\mathrm{Var}(\tilde{r}) > \mathrm{Var}(\tilde{p}) \tag{4.69}$$

One possible explanation for this result lies in the fact that the monopolist responds to the stochastic changes of the demand by adjusting

both variables under his control, the price p and the supply s, while the SHM dealer adjusts only one variable, the rental rate, r. Therefore, the rental rate r is more volatile compared with the new goods price p.

Furthermore, comparison of the rental rate and the inventory variances shows that

$$\mathrm{Var}(\tilde{x}) > \mathrm{Var}(\tilde{r})$$

if $|d_1| \geq 1$. In other words, the volatility of the rental rate will be smaller than the volatility of the inventory if the slope of the "adjusted"* excess demand for used goods is greater or equal to one. For a slope smaller than one, the comparison is inclusive, but if the slope approaches zero, the inequality will be reversed.

The stationary distribution of the supply is obtained by substituting (4.63) into (4.62) and letting $\tilde{s} = s^*(\tilde{x})$. Thus,

$$\tilde{s} = s_e - \frac{a_1 N_1}{2D} \sum_{k=0}^{\infty} \left(\frac{2d_1}{D}\right)^k u_{2k}.$$

The expected value, variance and covariances of the monopolist's supply are

$$E(\tilde{s}) = s_e \qquad , \ \mathrm{Var}(\tilde{s}) = \frac{a_1^2 N_1}{2d_1 D} \sigma_2^2 \tag{4.70}$$

$$\mathrm{Cov}(\tilde{x},\tilde{r}) = \frac{a_1 D}{d_1 N_1} \sigma_2^2 \ , \ \mathrm{Cov}(\tilde{s},\tilde{p}) = - \frac{a_1^2 N_1}{2a_2 d_1 D} \sigma_2^2$$

The relation between the monopolist's price and supply variances is

$$\mathrm{Var}(\tilde{p}) = \frac{1}{2a_2^2} \mathrm{Var}(\tilde{s})$$

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*I call the expected excess demand $g_2(r) = d_0 + d_1 r$, "adjusted" because the monopolist's price reaction function is already used to eliminate the price p.

This means that the price variance will be greater, equal or smaller than the supply variance depending upon whether the slope of the demand for new goods with respect to their price, i.e., a_2, is larger, equal or smaller than the negative square root of 0.5 correspondingly. In other words, as the slope of price with respect to expected demand becomes sufficiently large, price will become more volatile than the supply and conversely.

The positive sign of the derived covariances means that all three choice variables always move in the same direction. This result may be a testable hypothesis for empirical tests.

Following the same method as above, it can be shown that the stationary distribution of the random variable $\tilde{y}$ which represents the new good's inventory is given by

$$\tilde{y} = y_e - \sum_{k=0}^{\infty} u_{1k}$$

Thus

$$E(\tilde{y}) = y_e \quad \text{and} \quad Var(\tilde{y}) = \sigma_1^2 \tag{4.71}$$

From (4.70), (4.71) and the fact that the quantity produced is equal to supply minus inventory, i.e., $q = s - y$, we can easily see that if $\tilde{q}$ is the random variable representing the quantity produced, it holds that

$$E(\tilde{q}) = q_e = s_e - y_e \quad \text{and} \tag{4.72}$$

$$Var(\tilde{q}) = Var(\tilde{s}) + Var(\tilde{y}) = \frac{a_1 N_1}{2d_1 D} \sigma_2^2 + \sigma_1^2$$

The above results provide a significant point with respect to the behavior of the quantity produced. The long time contention of the production smoothing literature is that firms keep inventories in order

to smooth their production. Here is shown that the variability of the production level is larger than the variability of the demand for new goods. Of course, somebody can contend that this is a result of the linearity of the production cost. The answer is that under a linear production cost we may expect the variability of production to be equal at most to the variability of the demand for new goods. The point here is that the variability of the demand of related goods will also have an effect upon the production. Thus, empirical tests that fail to detect output smoothing behavior (Blinder (4)) do not discredit the inventory models, because these models do not predict production smoothing, especially in the case where more than one firm is included in the market system.

Concluding Remarks

The analysis of the present chapter makes evident that the quest for more realism in economic models leads to highly complex situations. The introduction of a stochastic framework without the market clearing requirement makes the analysis so difficult that a number of simplifications are required to keep the problem manageable. In particular, I had to assume linear demand functions, two period durability and linear production costs, in order to be able to derive exact expressions for the optimal strategies in the durable goods market system. The most serious simplification was the production cost linearity. As it is known from the literature, this assumption leads to constant price policies. Did this assumption make our results too unrealistic? The answer depends upon the perspective. One of the purposes of the analysis was to highlight the relationship between the new and used

goods markets. This purpose was accomplished. The derived optimal solution shows how the parameters and variables of each market affect the other. The linearity of the production cost resulted in the independence of the optimal solution, the trading process and the stantionary distributions of the variables of the system from the level for the new goods' inventory. This drawback can be corrected by a future research project that will use a more general production cost function. Of course, this analysis will be much more difficult.

Let's now compare the results of the the present analysis with the previous literature. The analysis confirms that argument advanced by Miller (13) and Benjamin and Kormendi (2), which says that a monopolist will always support the existence of a SHM, in order to realize higher prices for the new goods. The derived price reaction function of the monopolist shows that the higher the rental rate the higher the price the monopolist can charge.

Expansion of the present model to the case of a good that lasts more than two periods is very difficult because this will increase the state variables (inventories of goods of different age should be treated separately) and markets (separate markets for used goods of different age) of the system. Intuition suggests that in this case, the inventory of all the used goods in the market will be relatively high, thus the demand of new goods will be lower forcing the monopolist to charge a lower price, closer to the competitive one. This intuitive result agrees with arguments advanced by Coase (7) and Swan (21).

As it was noted in the previous section, the analysis derives some results that can be empirically tested. In particular, the positive relation among all the choice variables and the negative relations

between each choice variable and the used goods inventory are testable hypotheses.

Finally, the results of the previous and present chapters suggest that when a durable good has finite durability and the horizon of the firms and consumers is infinite, questions about time inconsistency (i.e., the monopolist lowering his price and flooding the market with new goods in the future in order to impose capital losses on the owners of used durable goods) do not arise.

CHAPTER FIVE

OPTIMAL BEHAVIOR OF THE SUPPLIERS IN THE NEW AND USED GOODS MARKET: THE COOPERATIVE CASE

Introduction

The purpose of the present chapter is to examine the optimal behavior of a durable goods monopolist who operates both the new and used goods markets. The analysis is equivalent to the case where a new goods monopolist and a SHM dealer are cooperating to maximize their total profit. Thus, the present chapter will examine the cooperative equilibrium of the system and will attempt some comparisons with the non-cooperative equilibrium derived in Chapter Four.

An attempt to describe the cooperative equilibrium in a durable goods market system was made by Swan (21) in his analysis of the ALCOA case. Arguing that the analysis of the case where the aluminum price changes between periods is very difficult, Swan examines the fixed price case. Even with fixed price a complete analysis is difficult, but Swan argues that from a welfare standpoint that a cooperative equilibrium (i.e., if ALCOA had integrated the aluminum recycling sector) is preferable to a non-cooperative equilibrium. The argument goes as follows: Given that both ALCOA and the competitive recycling sector charge the same price, we will expect that ALCOA will be producing at a lower Marginal Cost (because they produce where MC = MR) than the recycling sector (which produces at MC = P). Thus, by integrating the recycling sector, ALCOA will substitute high marginal cost recycled

aluminum with lower marginal cost virgin aluminum, increasing, thus, the welfare of the society.

The numerous difficulties that Swan faced in trying to derive the cooperative equilibrium became intensified when an uncertainty non-market clearing framework is used. Nevertheless, in what follows I will attempt the derivation of the interior solution, without going into details about the conditions required for the existence of this solution.

In what follows, the second section states the problem, the third section proves the existence, uniqueness and concavity of the value function, the fourth section derives the interior solution of the system, and also examines the equilibrium and adjustment properties of the system. In the fifth section, I examine the stationary distributions of the variables of the system, and finally the concluding remarks are given in the final section.

The Structure of the Problem

To facilitate comparisons with the previously obtained results, I will use the same demand and cost structure as in the previous chapter. In particular, I assume that the monopolist faces a linear stochastic demand for his new products and a linear stochastic excess demand for his used products. From the cost side, he faces a linear production cost function and quadratic holding-backlogging cost functions for new and used goods. The restrictions required to guarantee the non-negativity of the demand apply. The profitability condition changes to

$$2L_1 - a_0 - (a_1-b_2)r < a_2c_1$$

i.e., the monopolist now has to take into account the effect that a change of the price p will have upon the expected revenue from the used goods market. This effect is given by $(b_2 r)$.

Under the above assumptions, the monopolist's functional equation for the infinite horizon problem will be

$$F(y,x) = \max_{s,p,r} \{pg_1(r,p) + rg_2(r,p) - c_1(s-y) - \int h(s-g_1(r,p)-u_1)d\Phi_1(u) - \int l(x-g_2(r,p)-u_2)d\Phi_2(u) + \alpha\int F(s-g_1(r,p)-u_1, x-g_2(r,p)-u_2)d\Phi_{12}(u)\} \tag{5.1}$$

subject to

$$0 \le p \le \bar{p}(r) \quad , \quad 0 \le r \le p \quad , \quad y \le s \tag{5.2}$$

Comparing functional equation (5.1) with the functional equation in the previous chapter, we can see that the number of state variables increased by 100% and the number of choice variables by 50%. Note that in last chapter's functional equation r was simply a parameter, while here x is a state variable.

In what follows, I examine first the existence, uniqueness and concavity of the value function. Next I derive the interior solution, i.e., the solution that satisfies the first order conditions of the problem with equality. Cases where the first order conditions are inequalities will not be examined.

Existence, Uniqueness and Concavity of the Value Function

Suppose initially that only one period is remaining in the horizon. The value function at the end of the horizon, F_0, describing the value of any remaining inventory or undelivered backlogging can be any concave

function of y and x. For simplicity, I assume that $F_0 = 0$. Thus, the functional form that describes the one period problem is

$$F_1(y_2,x_2) = \max_{s_1,p_1,r_1} G_1(p_1,s_1,r_1,y_2,x_2) \tag{5.3}$$

with

$$\begin{aligned} G_1(p_1,s_1,r_1,y_2,x_2) = {} & p_1g_1(r_1,p_1) + r_1g_2(r_1,p_1) \\ & - c_1(s_1-y_2) - \int h(s_1-g_1-u_1)d\Phi_1(u) \\ & - \int l(x_2-g_2-u_2)d\Phi_2(u) \end{aligned} \tag{5.4}$$

Note that the subscripts of the choice and state variables signify number of periods, while the subscripts of g's and u's refer to the new or used goods.

Inspection of G_1 reveals that it is concave in s_1,y_2,x_2 but not necessarily so in r_1 and p_1. The reason is the nonstrict concavity of the term p_1r_1 which is included in the expected return function. We will show below that when the condition

$$4a_2b_1 \geq (a_1+b_2)^2 \tag{5.5}$$

holds, then G_1 is concave in all its elements. In words, condition (5.5) means that the term p_1r_1 is outweighted by the concave terms of the expected revenue function.

Differentiating G_1 twice with respect to p_1 and r_1 using the quadratic holding-backlogging cost specifications, I obtain

$$\begin{aligned} D_{rr}G_1 &= 2b_1 - a_1^2h_2 - b_1^2l_2 < 0 \\ D_{pp}G_1 &= sa_2 - a_2^2h_2 - b_2^2l_2 < 0 \\ D_{rp}G_1 &= a_1 + b_2 - a_1a_2h_2 - b_1b_2l_2 > 0 \end{aligned}$$

and

$$\begin{aligned} D_{rr}G_1D_{pp}G_1 - (D_{rp}G_1)^2 = {} & 4a_2b_1 - (a_1+b_2)^2 + a_2b_1 -{}_1b_2)_{212}h_2 \\ & - 2(a_2b_1-a_1b_2)(a_2h_2+b_1l_2) \end{aligned}$$

The concavity requires $D_{rr}G_1 < 0$, $D_{pp}G_1 < 0$, $D_{rr}G_1D_{pp}G_1 - (D_{rp}G_1)^2 > 0$. Under condition (5.5) and given $(a_2b_1-a_1b_2)$ has been assumed positive, the above inequalities hold. Furthermore, in addition to the above G_1 has the following second derivatives

$$D_{ss}G_1 = -h_2 < 0 , \quad D_{sp}G_1 = a_2h_2 < 0 , \quad D_{sr}G_1 = a_1h_2 > 0 ,$$
$$D_{xx}G_1 = -l_2 < 0 , \quad D_{xp}G_1 = b_2l_2 > 0 , \quad D_{xr}G_1 = b_1l_2 < 0 ,$$
$$D_{yy}G_1 = D_{yp}G_1 = D_{yr}G_1 = D_{ys}G_1 = D_{yx}G_1 = 0$$

Through straightforward but tedious calculations, we can show that the principal minors of the Hessian of G_1 alternate signs. This means that G_1 is jointly concave in all its arguments.

Let us now introduce another condition which will prove to be helpful in determining signs later on. This condition is

$$\min(|a_2|,|b_1|) \geq \max(a_1,b_2), \tag{5.6}$$

in other words the cross-effect demand parameters (a_1,b_2) are smaller than both own-effect demand parameters (a_2,b_1) in absolute value. Condition (5.6) implies condition (5.5) but not otherwise.

From the concavity of the maximand G_1 in all its arguments and appealing to Iglehart's Lemma 1 (10), the functional form $F_1(y_2,x_2)$ can be shown to be concave in both its arguments. Continuation of the same line of arguments can show that for any t-period horizon the value function $F_t(y_{t1},x_{t+1})$ is concave in both its elements. Next, by appeal to arguments advanced by Bellman (1) or Denardo (8) it can be shown that an infinite horizon value function $F(y,x)$ exists, is unique and is the limit toward which $F_t(y_{t+1},x_{t+1})$ converges uniformly as t goes to infinity. Also, $F(y,x)$ satisfies the infinite horizon functional equation. Value function $F(y,x)$ is also concave in y and x, due to the concavity of $F_t(y_{t+1},x_{t+1})$ and its uniform convergence.

The Interior Solution

Differentiation of (5.1) with respect to the choice variables s,p,r assuming that constraints (5.2) are not effective, gives

$$D_sG(s^*,p^*,r^*,y,x) = 0 \tag{5.7}$$
$$D_pG(s^*,p^*,r^*,y,x) = 0$$
$$D_rG(s^*,p^*,r^*,y,x) = 0$$

with

$$D_sG(s,p,r,y,x) = -c_1 - [\int h'(.)d\Phi_1(u)-\alpha\int F_y(.,.)d\Phi_{12}(u)] \tag{5.8}$$

$$\begin{aligned} D_pG(s,p,r,y,x) = {} & g_1 + pg_{1p} + rg_{2p} + g_{1p}[\int h'(.)d\Phi_1(u) \\ & - \alpha\int F_y(.,.)d\Phi_{12}(u)] + g_{2p}[\int l'(.)d\Phi_2(u) \\ & - \alpha\int F_x(.,.)d\Phi_{12}(u)] \end{aligned} \tag{5.9}$$

$$\begin{aligned} D_rG(s,p,r,y,x) = {} & g_2 + rg_{2r} + pg_{1r} + g_{1r}[\int h'(.)d\Phi_1(u) \\ & - \alpha\int F_y(.,.)d\Phi_{12}(u)] + g_{2r}[\int l'(.)d\Phi_2(u) \\ & - \alpha\int F_x(.,.)d\Phi_{12}(u)] \end{aligned} \tag{5.10}$$

where the suppressed arguments of the functions h,l and F are the same as in (5.1). In addition F_y and F_x are the derivatives of F with respect to its first and second argument correspondingly.

To obtain specific expressions for the optimal values s*, p*, r* the form of the value function F must be explicit. I claim that the general form of the value function will be

$$F(y,x) = \Theta_s y + \Theta_6 x + \frac{\pi_7}{2} x^2 + \rho_5 \tag{5.11}$$

The intuition for this claim is the following. As it is well known in the inventory literautre, in the backlogging case, the value function inherits the properties of the production cost function. Thus, under the assumption that the production cost of new goods is linear the term

including the inventory of new goods will be linear. Furthermore, as it was shown in the previous chapter the value function inherits the properties of the inventory cost function for the good not currently produced. Therefore, the value function will be quadratic with respect to the used goods inventory. (Later I will try to prove the above claim.)

Returning now to our problem, I can rewrite (5.7) using (5.8)-(5.11) and the specifications of the cost functions as

$$D_sG(s^*,p^*,r^*,x,y) = \alpha\Theta_5 - h_1 - c_1 - h_2(s^*-g_1^*) = 0 \tag{5.12}$$

$$D_pG(s^*,p^*,r^*,x,y) = g_1 + a_2p^* + b_2r^* - a_2c_1 - b_2[(\alpha\Theta_6-l_1) + (\alpha\pi_7-l_2)(x-g_2^*)] = 0 \tag{5.13}$$

$$D_rG(s^*,p^*,r^*,x,y) = g_2^* + b_1{}^* + a_1p^* - a_1c_1 - b_1[(\alpha\Theta_6-l_1) + (\alpha\pi_7-l_2)(x-g_2^*)] = 0 \tag{5.14}$$

where: $g_1^* = a_0 + a_1r^* + a_2p^*$, $g_2^* = b_0 + b_1r^* + b_2p^*$. Note that I already substituted (5.12) into (5.13) and (5.14).

Division of (5.13) and (5.14) and solving with respect to p^* yields:

$$p^*(r^*) = \frac{(b_0b_2-a_0b_1) - (a_2b_1-a_1b_2)c_1 - b_1(a_1-b_2)r^*}{2(a_2b_1-a_1b_2) + b_2(a_1-b_2)}. \tag{5.15}$$

This relation between the optimal price and the optimal rental rate corresponds to the monopolist's price reaction function in the non-cooperative case. Closer observation reveals that for $b_2 = 0$ the above relation is the same as relation (4.28) of the previous chapter. The size of the parameter b_2 is significant in order to determine the relation between p and r. Assuming that condition (5.6) holds, for b_2 smaller than a_1 the relation between p and r is positive, for $b_2 = a_1$,

price p becomes independent of r, while for b_2 larger than a_1 the relation becomes negative.

The results are evident from

$$\frac{\partial p^*(r^*)}{\partial r^*} = \frac{-b_1(a_1-b_1)}{2(a_2b_1-a_1b_2) + b_2(a_1-b_2)} \tag{5.16}$$

Therefore, a first remark of the analysis is that the present examination of a cooperative solution is more complicated than the non-cooperative analysis of the previous chapter. In particular, the parameters a_1 and b_2 that measure the responsiveness of each demand function to changes in the price in the other market, i.e., the cross-effects, are very significant in making price decisions.

Let us proceed now to determine the optimal values for s, p and r. Solving the system (5.12)-(5.14), after tedious calculations, I obtain

$$p^*(x) = \frac{M_5 + M_6x}{\Delta} \tag{5.17}$$

$$r^*(x) = \frac{N_5 + N_6x}{\Delta} \tag{5.18}$$

$$s^*(x) = \frac{(\alpha\Theta_6-h_1-c_1)\Delta+h_2(a_0\Delta+a_1N_5+a_2M_5)+h_2(a_1N_6+a_2M_6)x}{h_2\Delta} \tag{5.19}$$

where

$$\begin{aligned}
M_5 &= (a_2c_1-a_0)[sb_1+b_1^2(\alpha\pi_7-1_2)] + a_1b_1(b_0-b_2c_1)(\alpha\pi_7-1_2) \\
&\quad - (a_1c_1-b_0)(a_1+b_2) + b_1(b_2-a_1)(\alpha\Theta_6-1_1) \\
M_6 &= b_1(b_2-a_1)(\alpha\pi_7-1_2) \\
N_5 &= (a_1c_1-b_0)[(2a_2_b_2^2(\alpha\pi_7-1_2)] - (a_2c_1-a_0)[a_1+b_2+b_1b_2(\alpha\pi_7-1_2)] \\
&\quad + (2a_2b_1-a_1b_2-b_2^2)[(\alpha\Theta_6-1_1)] \\
N_6 &= (2a_2b_1-a_1b_2-b_2^2)(\alpha\pi_7-1_2) < 0 \\
\Delta &= 2b_1(a_2b_1-a_1b_2)(\alpha\pi_7-1_2) - (a_1+b_2)^2 + 4a_2b_1 > 0
\end{aligned}$$

Assuming that parameter π_7 is negative (which is consistent with the concavity of the value function), the denominator in all three optimal values is positive, and N_6 is negative. Direct comparison of the cooperative solution derived in (5.17)-(5.19) with the non-cooperative solution of the previous chapter is very difficult. One interesting area of comparison is the responses of the optimal policies to inventory changes. Differentiation of (5.17)-(5.19) with respect to x yields:

$$\frac{\partial p^*(x)}{\partial x} = \frac{M_6}{\Delta} \tag{5.20}$$

$$\frac{\partial r^*(x)}{\partial x} = \frac{N_6}{\Delta}$$

$$\frac{\partial s^*(x)}{\partial x} = \frac{a_1N_6+a_2M_6}{\Delta} = \frac{(a_2b_1-a_1b_2)(a_1+b_2)(\alpha\pi_7-1_2)}{\Delta},$$

$$0 > \frac{\partial s^*(x)}{\partial x} > -1$$

Given that condition (5.6) holds, N_6 is negative so that r^* varies inversely with inventory. Also the supply of new goods s^* varies inversely with invertory x, but given that $\frac{\partial s^*(x)}{\partial x} > -1$, the supply of new goods falls less than the increase in inventory x. The sign of the price p^* response to inventory depends upon the sign of the difference (b_2-a_1). That is, if a_1 is greater than b_2, then the price will have an inverse relationship with the inventory, if $a_1 = b_2$, then the price of new goods is independent of the used goods inventory and finally for a_1 less than b_2 price and inventory vary in the same direction.

As it is obvious from the above analysis the main difference between the cooperative and the non-cooperative solutions lies in the relation of the new goods price to the used goods inventory. In

particular, for a_1 less than b_2 the two solutions predict different responses. An explanation of the positive relation between price and inventory in this last case, for the cooperative solution, is in order. In the non-cooperative solution, an increase, say, in the inventory of used goods x will force the SHM dealer to lower his price r. Lower r will lead to lower p by the monopolist in order to prevent too many customers shifting to the SHM. In the present cooperative case, reactions may be different. An increase in x will lead to a lower r, but now it may not be profitable for the monopolist to lower p in order to prevent customer shifting toward used goods. Specifically, if the parameter a_1 is smaller than b_2, customers shift more easily from the SHM to the new goods market than otherwise. Thus, the monopolist, trying to reduce the used goods inventory x, will encourage customer shifting by increasing his price p.

Let us now examine the equilibrium of the system as we did in the previous chapter. Using the definition that equilibrium inventory is that level of inventory for which next period's expected inventory is equal to current inventory, I obtain the following:

$$y_e = s^* - g_1^* = \frac{\alpha\Theta_6 - h_1 - c_1}{h_2} \tag{5.21}$$

$$x_e = x_e - g_2^*(x_e) = -\frac{b_0\Delta + b_1N_5 + b_2M_5}{b_1N_6 + b_2M_6} \tag{5.22}$$

The equilibrium values for the choice variables are readily obtainable by substituting x_e for x in the formulas (5.17), (5.18) and (5.19). Defining $p_e = p^*(x_e)$, $r_e = r^*(x_e)$ and $s_e = s^*(x_e)$, we have

$$p_e = \frac{b_1(M_5N_6 - N_5M_6) - b_0M_6\Delta}{\Delta(b_1N_6 + b_2M_6)}$$

$$r_e = \frac{b_2(N_5M_6-M_5N_6)-b_0N_6\Delta}{\Delta(b_1N_6+b_2M_6)}$$

$$s_e = \frac{(a_2b_1-a_1b_2)(N_6M_5-N_5M_6)-b_0\Delta(a_1N_6+a_2M_6)}{\Delta(b_1N_6+b_2M_6)} + a_0 + \frac{\alpha\Theta_6-h_1-c_1}{h_2}$$

Using the above equilibrium formulas and the optimal formulas given in (5.17), (5.18), (5.19) we arrive at the following adjustment expressions

$$x - g_2^* = x_e + \frac{4a_2b_1-(a_1+b_2)^2}{\Delta}(x-x_e) \tag{5.23}$$

$$p^*(x) = p_e + \frac{M_6}{\Delta}(x-x_e) \tag{5.24}$$

$$r^*(x) = r_e + \frac{N_6}{\Delta}(x-x_e) \tag{5.25}$$

$$s^*(x) = s_e + \frac{a_1N_6+a_2M_6}{\Delta}(x-x_e)$$

From (5.23) we can show that the expected inventory adjusts partially towards its equilibrium value. The adjustment coefficient is positive and smaller than one so that the expected inventory will be lower (higher) than the current inventory if current inventory is higher (lower) than its equilibrium value. The signs and size of the adjustment coefficients for the price p and rental rate r are not immediately determined. The adjustment coefficient of the price p will be positive, zero or negative according to the sign of the difference (b_2-a_1). In particular, for b_2 greater than a_1 the price adjustment coefficient will be positive and smaller than one (partial adjustment), for b_2 equal to a_1 the adjustment coefficient will be zero and the price will remain always at its equilibrium level while for a_1 greater than b_2, the

adjustment coefficient will be negative and we are unable to determine if it is above or below minus one. Thus, in this last case the new goods price in a cooperative equilibrium will move in the opposite direction than it does in the non-cooperative case. Next, examination of the adjustment coefficient of the rental rate r reveals that the coefficient has the expected negative sign but again the size cannot be determined precisely. We can show that when a_1 and b_2 are very small the coefficient will be between zero and minus one. Finally, as we showed in examining the derivatives the adjustment coefficient for the supply is negative and larger than minus one, implying that the supply of the new durable goods will adjust partially towards a direction opposite to $(x-x_e)$.

Thus, the above analysis reinforces the previously mentioned points of difference between the cooperative and the non-cooperative solutions. In particular, in the cooperative solution the relative size of the cross-effect demand coefficients b_2 and a_1 becomes very crucial in the sense that it may cause price movements in the opposite direction than predicted under the non-cooperation assumption.

Stationary Distributions

As in the previous chapter, we can now derive the stationary distributions for the variables of the problem. This derivation as well as all the previous results are based on the assumption that the solution of the system is always interior. Again, we should notice that the linearity of the production cost function implies that the optimal values of the choice variables will be independent of the new good's inventory y and consequently the stationary distributions of the other

variables will be independent of the new good's demand disturbances. I start first by deriving the stationary distribution for the used good's inventory x.

Let $\tilde{x}_t$ be the random variable that represents the used good's inventory. At the beginning of the horizon the inventory takes the value of x_0. After the first period trading takes place, the used good's inventory becomes

$$\begin{aligned} x_1 &= x_0 - g_2(r^*(x_0),p^*(x_0)) - u_{20} \\ &= x_0 - b_0 - b_1 \frac{N_5+N_6x_0}{\Delta} - b_2 \frac{M_5+M_6x_0}{\Delta} - u_{20} \end{aligned}$$

or, by using the formula for x_e from (5.22)

$$\tilde{x}_1 = (1-K)x_0 + Kx_e - u_{20}$$

where $$K = \frac{b_1N_6+b_2M_6}{\Delta}.$$

After two periods of trading we have

$$\begin{aligned} \tilde{x}_2 &= \tilde{x}_1 - g_2(r^*(\tilde{x}_1),p^*(\tilde{x})) - u_{21} \\ &= (1-K)^2x_0 + (1-K)(Kx_e-u_{20}) + (Kx_e-u_{21}). \end{aligned}$$

And after t period of trading

$$\tilde{x}_t = (1-K)^tx_0 + \sum_{j=0}^{t-1} (1-K)^{t-(j+1)}(Kx_e-u_{2j}).$$

By letting $k = t-(j+1)$, we can rewrite $\tilde{x}_t$ as

$$\tilde{x}_t = (1-K)^tx_0 + \sum_{k=0}^{t-1} (1-K)^k(Kx_e-u_{2t-k-1}).$$

Since u_{2t} has been assumed identically distributed and independent of the paramaters of the problem, random variable $\tilde{x}_t$ satisfies

$$\tilde{x}_t = (1-K)^tx_0 + \sum_{k=0}^{t-1} (1-K)^k(Kx_e-u_{2k}).$$

Given that x_e is non-random and since we can write $K = 1-(1-K)$, $\tilde{x}_t$ can be written as

$$\tilde{x}_t = (1-K)^t x_0 + [1-(1-K)^t]x_e - \sum_{k=0}^{t-1} (1-K)^k u_{2k}$$

Taking now limits as t goes to infinity, we obtain

$$\tilde{x} = x_e - \sum_{k=0}^{\infty} (1-K)^k u_{2k} \tag{5.27}$$

Since we assumed that u_2 has expected value zero and variance σ_2^2, equation (5.27) implies that

$$E(\tilde{x}) = x_e \;,\; Var(\tilde{x}) = \frac{1}{1-(1-K)^2}\sigma_2^2 \tag{5.28}$$

$$= \frac{\Delta^2}{\Delta^2-[4a_2b_1-(a_1+b_2)^2]^2}\sigma_2^2.$$

From (5.28) it is obvious that the variance of the inventory x will be smaller than the variance of the stochastic term u_2. This result differs sharply with the non-cooperative case, where the used goods' inventory variance is larger than the stochastic term variance. The explanation of this difference lies in the fact that in the cooperative case the price and supply decisions of the monopolsit will have to be made taking into account the effort to minimize the used goods inventory cost, while in the non-cooperative case, the monopolist has no interest in such a cost minimization.

Substitution of the stationary distribution derived in (5.27) into the choice variables' adjustment equations (5.24)-(5.26) will give the following stationary distributions for the random variables $\tilde{p} = p^*(\tilde{x})$, $\tilde{r} = r^*(\tilde{x})$ and $\tilde{s} = s^*(\tilde{x})$

$$\tilde{p} = p_e + \frac{M_6}{\Delta}\sum_{k=0}^{\infty} (1-K)^k u_{2k} \tag{5.29}$$

$$\tilde{r} = r_e + \frac{N_6}{\Delta} \sum_{k=0}^{\infty} (1-K)^k u_{2k} \tag{5.30}$$

$$\tilde{s} = s_e + \frac{a_1N_6+a_2M_6}{\Delta} \sum_{k=0}^{\infty} (1-K)^k u_{2k} \tag{5.31}$$

Also, from (5.29) and (5.30) we can derive the relationship between $\tilde{p}$ and $\tilde{r}$

$$\tilde{p} = p_e + \frac{m_6}{M_6} (\tilde{r}-r_e). \tag{5.32}$$

The expected values, variances and covariances of the abhove stationary distributions are

$$E(\tilde{p}) = p_e \quad , \quad Var(\tilde{p}) = \left(\frac{M_6}{\Delta}\right)^2 Var(\tilde{x}) \tag{5.33}$$

$$E(\tilde{r}) = r_e \quad , \quad Var(\tilde{r}) = \left(\frac{N_6}{\Delta}\right)^2 Var(\tilde{x}) \tag{5.34}$$

$$E(\tilde{s}) = s_e \quad , \quad Var(\tilde{s}) = \left(\frac{a_1N_6+a_2M_6}{\Delta}\right)^2 Var(\tilde{x}) \tag{5.35}$$

$$Cov(\tilde{p},\tilde{r}) = E(\tilde{p}-p_e)(\tilde{r}-r_e) = \frac{M_6N_6}{\Delta^2} Var(\tilde{x})$$

$$Cov(\tilde{p},\tilde{s}) = E(\tilde{p}-p_e)(\tilde{s}-s_e) = \frac{(a_1N_6+a_2M_6)M_6}{\Delta^2} Var(\tilde{x})$$

$$Cov(\tilde{r},\tilde{s}) = E(\tilde{r}-r_e)(\tilde{s}-s_e) = \frac{(a_1N_6+a_2M_6)N_6}{\Delta^2} Var(\tilde{x})$$

Comparison of the variances of the price and the rental rate shows that:

$$Var(\tilde{r}) < Var(\tilde{p})$$

because $N_6 < M_6$. This result is opposite to the one reached in the non-cooperative case, showing that in the cooperative case the monopolist

will face the uncertainty in the market by varying the new good's price more than the rental rate.

From (5.35) and (5.20) we can see a "supply smoothing" effect, i.e.,

$$\mathrm{Var}(\tilde{s}) < \mathrm{Var}(\tilde{x}).$$

As I pointed out in discussing the adjustment properties, the values of the ratios $\frac{M_6}{\Delta}$ and $\frac{N_6}{\Delta}$ will depend upon the size of the demand parameters and we cannot determine in advance if they are higher or lower than one (in absolute value). Therefore, we cannot say anything about "price smoothing."

Finally, defining $\tilde{y}$ as the random variable representing the inventory and following the same inductive method as for $\tilde{x}$, we can easily derive that

$$\tilde{y} = y_e + \sum_{k=0}^{\infty} u_{1k} \tag{5.36}$$

The above equation is the stationary random variable for the new goods inventory. From the assumption that u_1 has expected value zero and variance σ_1^2 we derive

$$E(\tilde{y}) = y_e \ , \ \mathrm{Var}(\tilde{y}) = \sigma_1^2.$$

Finally, defining $\tilde{q}$ to be the random variable representing the new goods production and given that $q^* = s^* - y$, we can derive

$$E(\tilde{q}) = s_e - y_e \text{ and } \mathrm{Var}(\tilde{q}) = \frac{(a_1N_6+a_2M_6)^2}{\Delta^2-[4a_2b_1-(a_1+b_2)^2]^2}\,\sigma_2^2 + \sigma_1^2. \tag{5.37}$$

The variance $\tilde{q}$ was obtained by using (5.28), (5.35) and the fact that $\tilde{s}$ is independent of $\tilde{y}$, so that their covariance is zero. The fact that the variance of q is larger than the variance of the stochastic demand for new goods shows that the model does not display "production

smoothing." In other words, effects from the market of substitutes (here the SHM) increase the variability of the production above the variance of the own demand.

Concluding Remarks

The main focus of the preceding analysis was to make comparisons between the cooperative and the non-cooperative solutions for the durable good's market system. Despite examining only the interior solution for the cooperative case, the results proved to be much more complicated than in the non-cooperative case, and therefore direct comparisons proved very difficult. Nevertheless, some comparisons were possible mainly with respect to reactions of the optimal values to changes of the used goods inventory. In particular, in the cooperative case the monopolist will use the new goods price to offset part of the used goods inventory changes, while in the non-cooperative case he did not have any incentive to do so. This use of the new goods price results in a smaller variability for the inventory and the rental rate. Furthermore, the changes of the new goods price with respect to the used goods inventory will depend upon the cross-effect demand parameters and there may be instances where the price movements in the cooperative case will be opposite to the ones predicted in the non-cooperative case.

CHAPTER SIX

SUMMARY

The introduction of uncertainty in the study of the behavior of consumers and suppliers in a new and used durable goods market system proved quite significant.

The demand function for new and used durable goods were derived based on the assumption of future price uncertainty. Consumer behavior was thus predicated on the basis of present prices and probabilistic expectations of future prices. The demand for new and used durable goods is then a function of current prices and wealth. In addition, under other plausible assumptions, the properties of these demand functions were derived. It was shown that the demand for new goods depends inversely upon its own price and directly upon the used goods price. Correspondingly, the demand for used goods depends inversely upon its own price and directly upon the new goods price. These results are different from properties of conventional demand functions derived under the certainty assumption and presented in previous studies that show the demand of a durable good as a function of the discounted value of its services during its lifetime.

The fact that under price uncertainty the demands for new and used goods are functions of the current prices only, enables us to study the optimal behavior of the suppliers in both the new and used goods markets. For the analysis of the suppliers behavior dynamic programming

techniques, developed to analyze the behavior of price and quantity setting firms under uncertainty, were used. The use of these techniques enabled relaxation of the market clearing assumption and highlighted inventory movements. This facilitated the derivation of equilibrium values, adjustment mechanisms and stationary distributions for the state and choice variables.

The optimal behavior of the suppliers, i.e., their price and quantity strategies, were examined in two different cases. In the first case, the monopolistic supplier of new durable goods has to compete against the SHM supplier, i.e., when there is no cooperation; and in the second case, the monopolist operates both markets, i.e., when the profit maximization is achieved through cooperation of the two markets.

Chapter Four examines the non-cooperative case and presents results under a number of simplifications. The more severe of these simplifications is the assumption that the new durable goods are produced under a linear cost function. The optimal strategies of the suppliers in both markets are functions of the inventory of used goods but independent of the new goods inventory. Specifically, higher (lower) used goods inventories in the SHM will result in lower (higher) prices for both new and used goods and lower (higher) supply of new goods. Furthermore, partial adjustment mechanisms, derived for the choice variables (prices and new goods supply), showed that these variables adjust opposite to the used goods inventory movements and toward their equilibrium values (equilibrium values can be derived by solving the corresponding certainty problem). Finally, the stationary distributions for the state and choice variables of the system were derived. From these distributions expected values and variances for the variables were also derived.

Comparison of the variances reveals that the SHM variables are more volatile than the new goods market choice variables.

In Chapter Five the cooperative case is examined and the results compared with those from the non-cooperative case. Due to the complexity of the cooperative case, some significant questions could not be answered. For example, the question "Under which case is the monopolist's profits higher?" could not be answered quantitatively even if intuition suggests the cooperative case. Nevertheless, it was shown that in the cooperative case the optimal strategies are functions of the used goods inventory but the direction of the dependence upon this inventory would depend upon the relative size of the demand cross-effect parameters. While the used goods price and new goods supply would depend inversely upon the used goods inventory, the price of new goods could be independent, directly dependent or inversely dependent upon the used good inventory. This would depend upon whether the used goods price effect on the demand of new goods is equal, smaller or larger than the new goods price effect upon the demand of used goods, respectively. Moreover, the partial adjustment mechanisms at this stage do not have definite signs because those too depend upon the relative size of the cross-effect parameters. Finally, derivation of the stationary distributions of the variables of the system revealed that now the variance of the new goods price is larger than the used goods price variance, inverting the non-cooperative case result.

A number of the above mentioned results could be empirically testable, for example, the negative relation between choice variables and used goods inventory in the non-cooperative case and the relation between the variances of new and used goods prices in both cases.

Several extensions of the derived results are possible at the expense of an increase in the level of complexity of the calculations. First, introduction of a quadratic cost function could make the optimal solutions and the adjustment mechanisms functions of the new goods inventory as well. The stationary distribution functions would then also depend upon disturbances in the new goods demand.

Second, the assumption that the durable good lasts only two periods could be relaxed in favor of a good lasting some finite number of periods at the expense of introducing more choice and state variables for used goods of different ages and, finally, a third extension would be the analysis of the case where the excess demand is lost instead of being backlogged.

APPENDIX A

This appendix derives the demand and excess demand functions for a consumer who faces positive probabilities of rationing in both markets for new and used durable goods. The structure of the problem remains the same as described in the main text. The consumer's decision process is different now. The consumer initially places orders for new and used goods and depending upon the realization of these orders, he then places orders for the perishable good A which is assumed in abundant supply. In other words, the order z_A is conditional upon the realization of the orders z_B and z_C.

Let Δ_2 and Δ_3 be indicators that take value 1 when the consumer's order is delivered and value 0 when the order is not delivered, i.e., the consumer is rationed. Assume next that δ_2 and δ_3 are the probabilities that positive orders z_B and z_C correspondingly are delivered. The probability of rationing is zero for order z_A, and for order z_C when it is negative, i.e., when the consumer decides to sell part of his inventory. This last assumption means that the SHM dealer(s) never refuses to buy any quantity at the going price. The above assumptions amount to

$$z_B(\Delta_2) = \begin{cases} z_B(1) = z_B & \text{with probability } \delta_2 \\ z_B(0) = 0 & \text{with probability } (1-\delta_2) \end{cases}$$

and

$$z_C(\Delta_3) = \begin{cases} z_C(1) = z_C & \text{with probability } \delta_3 \\ z_C(0) = \min(0,z_C) & \text{with probability } (1-\delta_3) \end{cases}$$

The expectation of any function g which contains z_B and z_C as arguments, is given by

$$\underset{\delta_2\delta_3}{E} g[z_B(\Delta_2),z_C(\Delta_3)] = \delta_2\delta_3 g(z_B,z_C) + (1-\delta_2)\delta_3 g(0,z_C) \tag{A.1}$$

$$+ \delta_2(1-\delta_3)g(z_B,\min(0,z_C))$$

$$+ (1-\delta_2)(1-\delta_3)g(0,\min(0,z_C)).$$

Under the assumptions about rationing the consumption now may be different than the orders. The consumer will derive utility from the quantities consumed. Let c_A,c_B,c_C be the quantities consumed of goods A,B,C. The consumer's utility function is

$$U(c) = c_A^{(1-\gamma)\lambda}(c_B+c_C)^{\gamma\lambda} \quad , \quad 0 < \lambda < 1 \quad , \quad 0 < \gamma < 1$$

The relation between consumption and order is

$$c_A = z_A \quad , \quad c_B = z_B(\Delta_2) \quad , \quad c_C = z_C(\Delta_3) + v_C$$

Using this relation, I can rewrite the utility in terms of orders as

$$U(z) = z_A^{(1-\gamma)\lambda}[z_B(\Delta_2)+z_C(\Delta_3)+v_C]^{\gamma\lambda}$$

Assume now that the consumer maximizes the discounted sum of his expected utility over a finite horizon and let t to be the number of periods remaining in the horizon. The consumer's value function satisfies the equation

$$f_t(w_t;p_t,v_{Ct}) = \max_{z_{At},z_{Bt},z_{Ct}} \{\underset{\delta_2,\delta_3}{E} [U(z) +$$

$$+ \underset{p_{t-1}}{E} f_{t-1}(w_{t-1};p_{t-1},v_{C,t-1})]\} \tag{A.2}$$

subject to

$$w_t \geq p_{At}z_{At} + p_{Bt}z_{Bt}(\Delta_2) + p_{Ct}[z_{Ct}(\Delta_3)+v_{Ct}]$$

$$z_{Ct}(\Delta_3) + v_{Ct} \geq 0$$

$$p_t,z_{At},z_{Bt} \geq 0$$

and $f_0 = 0$,

with $p_t = (p_{At},p_{Bt},p_{Ct})$. The constraints are similar to the ones used in the main text.

The One Period Problem

Suppose now that only one period remains in the horizon. Developing the expectation operator as in (A.1), the functional form of the value function for the one period problem becomes

$$f_1(w_1;p_1,v_{C1}) = \max_{z_A,z_{B1},z_{C1}} \{z_{A1}^{(1-\gamma)\lambda}[\delta_2\delta_3(z_{B1}+z_{C1}+v_{C1})^{\gamma\lambda} + (1-\delta_2)\delta_3(z_{C1}+v_{C1})^{\gamma\lambda} + \delta_2(1-\delta_3)(z_{B1}+\min(0,z_{C1}) + v_{C1})^{\gamma\lambda} + (1-\delta_2)(1-\delta_3)(\min(0,z_{C1}) + v_{C1})^{\gamma\lambda}]\} \quad (A.3)$$

subject to

$$w_1 \geq p_{A1}z_{A1} + p_{B1}z_{B1} + p_{B1}(z_{C1}+v_{C1})$$

$$z_{C1} + v_{C1} \geq 0$$

$$p_1, z_{A1}, z_{B1} \geq 0$$

with

$$p_1 = (p_{A1}, p_{B1}, p_{C1}).$$

Furthermore, assume that the price of the new durable goods is higher than the price of the used durable goods, i.e., $p_B > p_C$, throughout the horizon.

The assumed Cobb-Douglas form of the utility function enables us to appeal to some well known standard results. First, the consumption of both goods should be positive. In the opposite case the total utility will be zero. Second, in the case of fully executed orders, i.e., when $\delta_2 = \delta_3 = 1$, the optimal solution, as shown in the main text, is

$$z^*_{A1} = \frac{(1-\gamma)w_1}{p_{A1}} \quad , \quad z^*_{B1} = 0 \quad , \quad z^*_{C1} = \frac{\gamma w_1}{p_{C1}} - v_{C1}$$

This solution provides some intuition about the solution in the case where rationing probabilities are positive. Specifically, I expect that the consumer will try to consume durable goods services valued at γw

because this will maximize his utility. When the consumer's inventory v_{C1} is higher than the desired value, then the consumer will sell quantity $z_{C1} = v_{C1} - \frac{\gamma w_1}{p_{C1}}$ and will consume the remaining quantity achieving thus the maximum utility level. If the consumer's inventory is less than $\frac{\gamma w_1}{p_{C1}}$ the consumer will place positive orders for (new or used) durable goods and will have to face the probability of rationing. Therefore, quantity $\frac{\gamma w_1}{p_{C1}}$ is a critical inventory level which I will call v^*_{C1}.

Let us now examine closer the case where the actual inventory or durable goods is below its critical level v^*_{C1}. If the consumer orders used goods to cover the difference, i.e., $z_{C1} = v^*_{C1} - v_{C1}$, then he faces the case where his order may not be executed with probability $(1-\delta_3)$. To insure himself against that probability the consumer may be willing to order some new goods. In this case the probability of not receiving any goods (new or used) falls to $(1-\delta_2)(1-\delta_3)$, but now the consumer may end up with more or less than the desired quantity depending on which order will be delivered. That is, now there is possibility that one or both orders will be delivered. Furthermore, the consumer will have to pay a higher price for the new goods that will be delivered. This suggests that when the difference $v^*_{C1} - v_{C1}$ is relatively small, the consumer will not order any new goods, while when the difference is large, there will be a level of inventory at which the expected marginal utility from the consumption of a new good will be larger than its price and the consumer will order new goods. Let v^{**}_{C1} be the inventory level below which the consumer will order positive quantities of both new and used goods. Recapitulating, the above intuitive analysis suggests the existence of two critical inventory levels such that, if the actual

inventory is above v^*_{C1}, the consumer will sell part of his inventory. If the actual inventory is between v^*_{C1} and v^{**}_{C1} the consumer will order only used goods, while when the actual inventory is below v^{**}_{C1} the consumer orders both new and used goods. In what follows, I proceed to prove the above claims and determine the critical inventory levels.

Let $\mu_{11} \geq 0$ and $\mu_{21} \geq 0$ be the multipliers for the constraints in (A.3). The Lagrangean of the problem is

$$L_1 = z_{A1}^{(1-\gamma)\lambda}(R_1) + \mu_{11}[w_1 - p_{A1}z_{A1} - p_{B1}z_{B1} - p_{C1}(z_{C1}+v_{C1})] \qquad (A.4)$$
$$+ \mu_{21}(z_{C1}+v_{C1})$$

where

$$R_1 = E_{\delta_2,\delta_3}[z_{B1}(\Delta_2)+z_{C1}(\Delta_3)+v_{C1}]^{\gamma\lambda}.$$

The corresponding first order conditions, taking into account that z_{A1} and $z_{C1}+v_{C1}$ must be positive and the consumer will spend all his wealth during this last period in the horizon, are

$$\frac{\partial L_1}{\partial z_{A1}} = (1-\gamma)\lambda\, z_{A1}^{(1-\gamma)\lambda-1} R_1 - \mu_{11}p_{A1} = 0 \qquad (A.5)$$

$$\frac{\partial L_1}{\partial z_{B1}} = \gamma\lambda z_{A1}^{(1-\gamma)\lambda}\delta_2[\delta_3(z_{B1}+z_{C1}+v_{C1})^{\gamma\lambda-1} + (1-\delta_3) \qquad (A.6)$$
$$(z_{B1}+\min(0,z_{C1})+v_{C1})^{\gamma\lambda-1}] - \mu_{11}p_{B1} \leq 0$$

$$\frac{\partial L_1}{\partial z_{C1}} = \gamma\lambda z_{A1}^{(1-\gamma)\lambda}\{\delta_2\delta_3(z_{B1}+z_{C1}+v_{C1})^{\gamma\lambda-1} + (1-\delta_2)\delta_3(z_{C1}+v_{C1})^{\gamma\lambda-1}$$
$$+ (1-\delta_3)[\delta_2(z_{B1}+\min(0,z_{C1})+v_{C1})^{\gamma\lambda-1} + (1-\delta_2)(\min(0,z_{C1})$$
$$+ v_{C1})^{\gamma\lambda-1}][\frac{\partial \min(0,z_{C1})}{\partial z_{C1}}\} - \mu_{11}p_{C1} + \mu_{21} = 0 \qquad (A.7)$$

$$\frac{\partial L}{\partial \mu_{11}} = w_1 - p_{A1}z_{A1} - p_{B1}z_{B1} - p_{C1}(z_{C1}+v_{C1}) = 0 \qquad (A.8)$$

$$\frac{\partial L}{\partial \mu_{21}} = z_{C1} + v_{C1} > 0. \qquad (A.9)$$

The last three conditions imply that $\mu_{11} > 0$ and $\mu_{21} = 0$ correspondingly. Examine first the case where $\min(0, z_{C1}) = z_{C1}$, i.e., the case that the consumer sells used goods, $z_{C1} < 0$.

Conditions (A.5), (A.6) and (A.7) become

$$\frac{\partial L_1}{\partial z_{A1}} = (1-\gamma)\lambda z_{A1}^{(1-\gamma)\lambda-1}[\delta_2(z_{B1}+z_{C1}+v_{C1})^{\gamma\lambda}+(1-\delta_2)(z_{C1}+v_{C1})^{\gamma\lambda}]$$

$$- \mu_{11}p_{A1} = 0 \tag{A.10}$$

$$\frac{\partial L_1}{\partial z_{B1}} = \delta_2\gamma\lambda z_{A1}^{(1-\gamma)\lambda}(z_{B1}+z_{C1}+v_{C1})^{\gamma\lambda-1} - \mu_{11}p_{B1} \leq 0 \tag{A.11}$$

$$\frac{\partial L_1}{\partial z_{C1}} = \gamma\lambda z_{A1}^{(1-\gamma)\lambda}[\delta_2(z_{B1}+z_{C1}+v_{C1})^{\gamma\lambda-1}+(1-\delta_2)(z_{C1}+v_{C1})^{\gamma\lambda-1}]$$

$$- \mu_{11}p_{C1} = 0. \tag{A.12}$$

From (A.11), (A.12) and the assumption $p_{B1} > p_{C1}$ we can easily see that $\frac{\partial L_1}{\partial z_{B1}} < 0$, which in turn implies that $z^*_{B1} = 0$. Therefore, the optimal solution is

$$z^*_{A1} = \frac{(1-\gamma)w_1}{p_{A1}} \quad , \quad (z_{C1}+v_{C1})^* = \frac{\gamma w_1}{p_{C1}} \tag{A.13}$$

Now define the critical inventory value v^*_{C1} such that when the actual inventory takes this critical value the used goods' order is zero or:

$$v^*_{C1} = \frac{\gamma w_1}{p_{C1}} \tag{A.14}$$

In order for the assumption $z^*_{C1} < 0$ (under which the system (A.10)-(A.12) was derived) to hold, the actual inventory must be greater than the critical inventory level. It can be shown (by contradiction) that when $v_{C1} < v^*_{C1}$ then z^*_{C1} must be positive.

Next, examine the case where z^*_{C1} is positive, i.e., when $v_{C1} < v^*_{C1}$. Positive z_{C1} means that $\min(0,z_{C1}) = 0$. Now the conditions (A.5)-(A.7) become

$$\frac{\partial L_1}{\partial z_{A1}} = (1-\gamma)\lambda z_{A1}^{(1-\gamma)\lambda-1}[\delta_2\delta_3(z_{B1}+z_{C1}+v_{C1})^{\gamma\lambda}+(1-\delta_2)\delta_3(_{C1}+v_{C1})^{\gamma\lambda}$$

$$+\delta_2(1-\delta_3)(z_{B1}+z_{C1})^{\gamma\lambda}+(1-W_2)(1-\delta_3)v_{C1}^{\gamma\lambda}]$$

$$-\mu_{11}p_{A1} = 0 \tag{A.15}$$

$$\frac{\partial L_1}{\partial z_{B1}} = \gamma\lambda z_{A1}^{(1-\gamma)\lambda}\delta_2[\delta_3(z_{B1}+z_{C1}+v_{C1})^{\gamma\lambda-1}+(1-\delta_3)(z_{B1}+v_{C1})^{\gamma\lambda-1}]$$

$$-\mu_{11}p_{B1} \leq 0. \tag{A.16}$$

$$\frac{\partial L_1}{\partial z_{C1}} = \gamma\lambda z_{A1}^{(1-\gamma)\lambda}\delta_3[\delta_2(z_{B1}+z_{C1}+v_{C1})^{\gamma\lambda-1}+(1-\delta_2)(z_{C1}+v_{C1})^{\gamma\lambda-1}]$$

$$-\mu_{11}p_{C1} = 0. \tag{A.17}$$

Examining condition (A.16) more closely, we can see that if z_{B1} is zero and the inventory v_{C1} gets very small, then the term $(z_{B1}+v_{C1})^{\gamma\lambda-1}$ becomes very large which means that the condition becomes positive. From the other side, the condition is required to be negative if the optimal value for z_{B1} is zero. Thus, here we have a contradiction. This contradiction means that when the inventory v_{C1} is very small then the optimal value of z_{B1} must be positive. We can determine how small the inventory should be in order for the consumer to order a positive quantity of new durable goods. In particular, we can determine a second critical inventory level v^{**}_{C1} such that, if the actual inventory is above v^{**}_{C1}, the consumer does not order new goods and if the actual inventory is below v^{**}_{C1} he orders positive quantities of new goods.

To determine v^{**}_{C1}, assume that $z^*_{B1} = 0$ and condition (A.16) holds with equality. Then the first order conditions can be written as

$$\frac{\partial L_1}{\partial z_{A1}} = (1-\gamma)\lambda z_{A1}^{(1-\gamma)\lambda-1}[\delta_3(z_{C1}+v_{C1})^{\gamma\lambda}+(1-\delta_3)v_{C1}^{\gamma\lambda}]$$

$$= \mu_{11}p_{A1}. \tag{A.18}$$

$$\frac{\partial L_1}{\partial z_{B1}} = \gamma\lambda z_{A1}^{(1-\gamma)\lambda}\delta_2[\delta_3(z_{C1}+v_{C1})^{\gamma\lambda-1}+(1-\delta_3)v_{C1}^{\gamma\lambda-1}]$$

$$= \mu_{11}p_{B1} \tag{A.19}$$

$$\frac{\partial L_1}{\partial z_{B1}} = \gamma\lambda z_{A1}^{(1-\gamma)\lambda}\delta_3(z_{C1}+v_{C1})^{\gamma\lambda-1} = \mu_{11}p_{C1}. \tag{A.20}$$

Division of (A.19) by (A.20) gives

$$\delta_2[1 + \frac{1-\delta_3}{\delta_3}(\frac{v_{C1}}{z_{C1}+v_{C1}})^{\gamma\lambda-1}] = \frac{p_{B1}}{p_{C1}}. \tag{A.21}$$

Solving (A.21) with respect to v_{C1} I obtain

$$v_{C1} = \frac{B}{1-B}\, z_{C1} \tag{A.22}$$

where

$$B = [\frac{\delta_3(p_{B1}-\delta_2 p_{C1})}{\delta_2(1-\delta_3)p_{C1}}]^{\frac{1}{\gamma\lambda-1}} \tag{A.22'}$$

Substituting now v_{C1} into (A.18) and (A.20) using (A.22) gives

$$\frac{\partial L_1}{\partial z_{A1}} = (1-\gamma)\lambda z_{A1}^{(1-\gamma)\lambda-1} z_{C1}^{\gamma\lambda}[\frac{\delta_3+(1-\delta_3)B^{\gamma\lambda}}{(1-B)^{\gamma\lambda}}] = \mu_{11}p_{A1}. \tag{A.18'}$$

$$\frac{\partial L_1}{\partial z_{C1}} = \delta_3\gamma\lambda z_{A1}^{(1-\gamma)\lambda} z_{C1}^{\gamma\lambda-1}(\frac{1}{1-B})^{\gamma\lambda-1} = \mu_{11}p_{C1}. \tag{A.20'}$$

Division of (A.18') by (A.20'), yields

$$\frac{(1-\gamma)z_{C1}}{\gamma\delta_3 z_{A1}}[\frac{\delta_3+(1-\delta_3)B^{\gamma\lambda}}{1-B}] = \frac{p_{A1}}{p_{C1}}. \tag{A.23}$$

Solving (A.23) with respect to $p_{A1}z_{A1}$ and substituting in the budget constraing (A.8), for $z_{B1} = 0$ and v_{C1} given in (A.22), I obtain

$$w_1 = p_{C1}z_{C1}\{1 + \frac{B}{1-B} + \frac{(1-\gamma)\lambda}{\gamma\delta_3}[\frac{\delta_3+(1-\delta_3)B^{\gamma\lambda}}{(1-B)}]\}.$$

or

$$z^{*}_{C1} = \frac{\gamma w_1}{p_{C1}} \left[\frac{3^{(1-B)}}{\delta_3+(1-\delta_3)(1-\gamma)B^{\gamma\lambda}}\right]. \tag{A.24}$$

Substitution of (A.24) into (A.22) will provide the second critical level of inventory, i.e.

$$v^{**}_{C1} = \frac{\gamma w_1}{p_{C1}} \left[\frac{3^{B}}{\delta_3+(1-\delta_3)(1-\gamma)B^{\gamma\lambda}}\right]. \tag{A.25}$$

When the actual inventory is below this second critical inventory level, the consumer will order positive quantities of both new and used goods, while if the actual inventory is larger or equal to the critical level the order of new durable goods will be zero.

Examine next the consumer's optimal solution when the actual inventory lies in the range between the two critical inventory levels. As it is shown above, in this range the new goods order is zero. The optimal values of the orders of good A and good C are determined by solving the three equation system (A.18), (A.20) and (A.8). Division of equation (A.18) by equation (A.20) and substitution of z_{A1} from (A.8) yields the expression

$$(z_{C1}+v_{C1})\left[1+(1-\gamma)\left(\frac{1-\delta_3}{\delta_3}\right)\left(\frac{v_{C1}}{v_{C1}+z_{C1}}\right)^{\gamma\lambda}\right] = \frac{\gamma w_1}{p_{C1}}.$$

Exact solution of this equation with respect to z_{C1} is very difficult. In principle, the above equation can be solved giving z_{C1} as a function of v_{C1}, w_1 and p_{C1}, i.e.

$$\tilde{z}_{C1} = z_{C1}(v_{C1}, w_1, p_{C1})$$

Therefore, when $v^{*}_{C1} > v_{C1} > v^{**}_{C1}$, the optimal solution is

$$\tilde{z}_{C1} = z_{C1}(v_{C1}, w_1, p_{C1}),\ \tilde{z}_{B1} = 0,\ \tilde{z}_{A1} = \frac{w_1 - p_{C1}(\tilde{z}_{C1}+v_{C1})}{p_{A1}}. \tag{A.26}$$

Note that the optimal value for the used goods order $\tilde{z}_{C1}$ will be positive, lying between zero (when the actual inventory equals v^*_{C1}) and the value given in (A.24) (when the actual inventory equals v^{**}_{C1}).

In the case where the actual inventory lies between zero and v^{**}_{C1}, the optimal values for the orders of all three goods are determined by solving the system (A.15)-(A.17) and (A.8), with condition (A.16) satisfied with equality. Due to the complexity of the system, derivation of the exact solution is very difficult, but it is easy to see that the optimal values will be functions of the wealth, prices and actual inventory level. That is

$$\hat{z}_{C1} = z_{C1}(w_1, p_{A1}, p_{B1}, p_{C1}, v_{C1})$$

$$\hat{z}_{B1} = z_{B1}(w_1, p_{A1}, p_{B1}, p_{C1}, v_{C1})$$

$$\hat{z}_{A1} = \frac{w_1 - p_{B1}\hat{z}_{B1} - p_{C1}(\hat{z}_{C1} + v_{C1})}{p_{A1}} \tag{A.27}$$

Substituting the optimal values obtained in (A.13), (A.26) and (A.27) in the value function (A.3) and defining

$$A(p_1) = [(1-\gamma)^{1-\gamma}\gamma^{\gamma}/(p_{A1}{}^{1-\gamma}p_{C1}^{\gamma})]^{\lambda}$$

I derive

$$\text{If } v_{C1} \geq v^*_{C1}\ , \text{ then } f_1(w_1;p_1,v_{C1}) = A(p_1)w_1{}^{\lambda} \tag{A.28}$$

$$\text{If } v^*_{C1} > v_{C1} \geq v^{**}_{C1}\ , \text{ then}$$

$$f_1(w_1;p_1,v_{C1}) = A(p_1)\{\delta_3[\frac{w_1 - p_{C1}(\tilde{z}_{C1}+v_{C1})}{1-\gamma}]^{(1-\gamma)\lambda}$$

$$[\frac{p_{C1}(\tilde{z}_{C1}+v_{C1})}{\gamma}]^{\gamma\lambda} + (1-\delta_3)(\frac{w_1 - p_{C1}v_{C1}}{1-\gamma})^{(1-\gamma)\lambda}$$

$$(\frac{p_{C1}v_{C1}}{\gamma})^{\gamma\lambda}\}$$

If $v_{C1}^{**} > v_{C1} \geq 0$, then

$$f_1(w_1;p_1,v_{C1}) = A(p_1)\{\delta_2\delta_3[\frac{w_1-p_{B1}\hat{z}_{B1}-p_{C1}(\hat{z}_{C1}+v_{C1})}{1-\gamma}]^{(1-\gamma)\lambda}$$

$$[\frac{p_{C1}(\hat{z}_{B1}+\hat{z}_{C1}+v_{C1})}{\gamma}]^{\gamma\lambda}+(1-\delta_2)\delta_3[\frac{w_1-p_{C1}(\hat{z}_{C1}+v_{C1})}{1-\gamma}]^{(1-\gamma)\lambda}$$

$$[\frac{p_{C1}(\hat{z}_{C1}+v_{C1})}{\gamma}]^{\gamma\lambda}+\delta_2(1-\delta_3)[\frac{w_1-p_{B1}\hat{z}_{B1}-p_{C1}v_{C1}}{1-\gamma}]^{(1-\gamma)\lambda}$$

$$[\frac{p_{C1}(\hat{z}_{B1}+v_{C1})}{\gamma}]^{\gamma\lambda}+(1-\delta_2)(1-\delta_3)(\frac{w_1-p_{C1}v_{C1}}{1-\gamma})^{(1-\gamma)\lambda}$$

$$(\frac{p_{C1}v_{C1}}{\gamma})^{\gamma\lambda}\}$$

Expression (A.28) gives the maximum expected utility for the one period horizon problem. Note that the use of the Cobb-Douglas utility function proved useful for the exact calculation of the two critical inventory levels, but too complicated for the exact calculation of all the optimal values.

The Two Period Problem

This section examines the optimal behavior of a consumer over a two period horizon. In what follows, period 2 is the first period of the horizon and period 1 is the last. In the beginning of period 2 the consumer's wealth consists of the value of the riskless bonds that he bought the previous period and the value of his inventory of durable goods, i.e.,

$$w_2 = \text{Value of Bonds} + p_{C2}v_{C2}.$$

After the consumer decides upon the allocation of his wealth, we have

$$w_2 = p_{A2}z_{A2} + p_{B2}z_{B2} + p_{C2}(z_{C2}+v_{C2}) + T_2 \tag{A.29}$$

where T_2 represents the value of riskless bonds the consumer wants to transfer to the next period. Given the assumption that the consumer

invests only in one-period bonds, T_2 is the value of bonds bought during period 2.

Let H_2 represent the consumer's savings in the end of period 2, defined as

$$H_2 = w_2 - p_{A2}z_{A2} - p_{C2}(z_{B2}+z_{C2}+v_{C2}) \tag{A.30}$$

The savings will be equal to the value of the consumer's portfolio which contains bonds and durable goods inventory. Specifically,

$$H_2 = T_2 + (p_{B2}-p_{C2})z_{B2}. \tag{A.31}$$

Next period's, i.e., period 1, wealth will be equal to portfolio's value at the beginning of that period, i.e.,

$$w_1 = rT_2 + p_{C1}v_{C1}$$

or, by using (A.31) to substitute for T_2 and given that $z_{B2} = v_{C1}$:

$$w_1 = rH_2 + [p_{C1}-r(p_{B2}-p_{C2})]z_{B2} \tag{A.32}$$

where $r = 1 + R$ is the return per dollar invested in the riskless bond.

The function that gives the consumer's maximum discounted sum of expected utility for the two-period horizon is

$$f_2(w_2;p_2,v_{C2}) = \max_{z_{A2},z_{B2},z_{C2}} \{\underset{\delta_2,\delta_3}{E} [U(z_2)+ \alpha \underset{p_1}{E} f_1(w_1;p_1,z_{B2})]\} \tag{A.33}$$

subject to

$$w_2 \geq p_{A2}z_{A2} + p_{B2}z_{B2}(\Delta_2) + p_{C2}[z_{C2}(\Delta_3)+v_{C2}] \tag{A.34}$$

$$z_{C2} + v_{C2} \geq 0$$

$$p_2,z_{A2},z_{B2},v_{C2} \geq 0$$

with w_1 given in (A.32).

Before I proceed in the solution of period 2's problem, we must calculate $\underset{p_1}{E} f_1(w_1;p_1,z_{B2})$ and solve the related portfolio problem.

Using (A.32) which describes period 1's wealth in terms of period 2's variables, I can rewrite the maximum expected utility for period 1 derived in (A.28) as

$$
\begin{aligned}
f_1(\cdot) &= A(p_1)g_1(\cdot)\ , \text{ when } v_{C1} \geq v^*_{C1} \text{ or} \\
&\quad [(1-\gamma)p_{C1}+\gamma r(p_{B2}-p_{C2})]z_{B2} \geq \gamma r H_2 \qquad \text{(A.35)} \\
f_1(\cdot) &= A(p_1)[\delta_3 g_2(\cdot)+(1-\delta_3)g_2(\cdot)] \quad \text{when } v^*_{C1} > v_{C1} \geq v^{**}_{C1} \text{ or} \\
&\quad [(\tfrac{1}{N} - \gamma)p_{C1}+\gamma r(p_{B2}-p_{C2})]z_{B2} \geq \gamma r H_2 > \\
&\quad [(1-\gamma)p_{C1}+\gamma r(p_{B2}-p_{C2})]z_{B2} \\
f_1(\cdot) &= A(p_1)[\delta_2\delta_3 g_4(\cdot)+(1-\delta_2)\delta_3 g_5(\cdot)+\delta_2(1-\delta_3)g_6(\cdot) \\
&\quad +(1-\delta_2)(1-\delta_3)g_7(\cdot)] \quad \text{when } v^{**}_{C1} > v_{C1} \geq 0 \quad \text{or} \\
&\quad 0 \geq \gamma r H_2 \geq [(\tfrac{1}{N} - \gamma)p_{C1}+\gamma r(p_{B2}-p_{C2})]z_{B2}
\end{aligned}
$$

where

$$f_1(\cdot) = f_1(rH_2+[p_{C1}-r(p_{B2}-p_{C2})]z_{B2};p_1,z_{B2})$$

$$g_i(\cdot) = g_i(H_2,p_1,p_{B2},p_{C2},z_{B2})\ , \ i = 1,2,\ldots,7$$

$$N = \frac{\delta_3 B}{\delta_3+(1-\delta_3)(1-\gamma)B^{\gamma\lambda}}$$

and B as given in (A.22'). Note that the inventory ranges were obtained by substituting (A.32) in the expressions (A.14) and (A.25).

Next, let us define expectation operators, with respect to period 1's prices, over the three inventory ranges, i.e., E^1 for the case where $v_{C1} \geq v^*_{C1}$, E^2 for $v^*_{C1} > v_{C1} \geq v^{**}_{C1}$ and E^3 for $v^{**}_{C1} > v_{C1}$. Let function G_1 satisfy

$$
\begin{aligned}
G_1(H_2,z_{B2},p_{B2}-p_{C2}) &= \underset{p_1}{E^1}A(p_1)g_1(\cdot) + \underset{p_1}{E^2}A(p_1)[\delta_3 g_2(\cdot) \\
&\quad +(1-\delta_3)g_3(\cdot)] + \underset{p_1}{E^3}A(p_1)[\delta_2\delta_3 g_4(\cdot) \\
&\quad +(1-\delta_2)\delta_3 g_5(\cdot)+\delta_2(1-\delta_3)g_6(\cdot) \\
&\quad +(1-\delta_2)(1-\delta_3)g_7(\cdot)] \qquad \text{(A.36)}
\end{aligned}
$$

In other words,

$$\underset{p_1}{E}\, f_1(rH_2+[p_{C1}-r(p_{B2}-p_{C2})]z_{B2};p_1,z_{B2}) = G_1(H_2,z_{B2},p_{B2}-p_{C2}). \qquad (A.37)$$

Function G_1 represents the maximum expected utility of the last period of the horizon, viewed from the beginning of period 2. The maximum expected utility depends directly upon the wealth and therefore upon the portfolio decisions that will be made during period 2. In what follows, I examine some of the properties of the function G_1 that will be useful in the subsequent analysis of the problem.

Due to the complexity of the problem, it is very difficult to prove the signs of the effects of changes in income and durable goods inventory upon period 1's maximum expected utility. Thus, I will proceed making some plausible claims upon these signs. In particular, I claim that the maximum expected utility of period 1 changes directly with wealth and inventory of durable goods, i.e.,

$$\frac{\partial f_1(w_1;p_1,z_{B2})}{\partial w_1} > 0 \quad \text{and} \quad \frac{\partial f_1(w_1;p_1,z_{B2})}{\partial z_{B2}} > 0.$$

The implication of the first expression is that

$$G_{1H}(H_2,z_{B2},p_{B2}-p_{C2}) > 0 \qquad (A.38)$$

where $\quad G_{1H}(\cdot) = \dfrac{\partial G_1(\cdot)}{\partial H_2}$

In other words, the maximum expected utility G_1 depends directly upon the amount of period 2's savings.

The effect of the durable inventory z_{B2} upon G_1 is more difficult to be derived. Differentiating G_1 with respect to z_{B2} we get

$$G_{1Z}(\cdot) = E\,\frac{\partial f_1(\cdot)}{\partial w_1}\,\frac{\partial w_1}{\partial z_{B2}} + E\,\frac{\partial f_1(\cdot)}{\partial z_{B2}}$$

But from (A.32) we can see that

$$\frac{\partial w_1}{\partial z_{B2}} = p_{C1} - r(p_{B2}-p_{C2})$$

whose sign is not determined, i.e., it can be positive or negative. By the same reasoning as in the main text, we can find an interval $[(p_{B2}-p_{C2})^m,(p_{B2}-p_{C2})^u]$ defined here by

$$G_{1Z}(H_2, \frac{H_2}{(p_{B2}-p_{C2})^m}, (p_{B2}-p_{B2})^m = 0$$

and

$$G_{1Z}(H_2,0,(p_{B2}-p_{C2})^u) = 0,$$

such that, when $(p_{B2}-p_{C2})$ lies in this interval, the optimal investment in durable goods is given by

$$G_{1Z}(H_2,z^*_{B2},(p_{B2}-p_{C2})) = 0. \tag{A.39}$$

Note that $z_{B2} = \dfrac{H_2}{p_{B2}-p_{C2}}$ means that the consumer invests all his savings in durable goods. Also, when $p_{B2}-p_{C2}$ is smaller than $(p_{B2}-p_{C2})^u$ then

$$G_{1Z}(H_2,0,p_{B2}-p_{C2}) > 0, \tag{A.40}$$

i.e., the consumer will be willing to invest part of his savings in durable goods.

In what follows, I proceed under the assumption that $p_{B2}-p_{C2}$ takes values in the interval $[(p_{B2}-p_{C2})^m,(p_{B2}-p_{C2})^u]$. In other words, I assume that the optimal portfolio solution is always interior, i.e., equation (A.39) always holds.

Let us now turn to the period 2 problem. Using the maximum expected utility for period 1 as derived in (A.37), we can rewrite (A.33) as

$$f_2(w_2;p_2,v_{C2}) = \max_{z_{A2},z_{B2},z_{C2}} \{E_{\delta_2,\delta_3} [z_{A2}^{(1-\gamma)\lambda}[z_{B2}(\Delta_2)+z_{C2}(\Delta_3)+v_{C2}]^{\gamma\lambda}$$
$$+\alpha G_1(H_2,z_{B2}(\Delta_2),p_{B2}-p_{C2})]\} \tag{A.41}$$

subject to (A.34).

The first constraint in (A.34) will be always ineffective, i.e., the rational consumer will not invest all his savings in the durable good. Furthermore, due to the Cobb-Douglas utility function, both z_{A2} and $z_{C2} + v_{C2}$ should be positive, which implies that the second constraint in (A.34) is also ineffective.

The first order conditions of the problem are

$$\frac{\partial f_2}{\partial z_{A2}} = (1-\gamma)\lambda z_{A2}^{(1-\gamma)\lambda-1}\{\delta_2\delta_3(z_{B2}+z_{C2}+v_{C2})^{\gamma\lambda}+(1-\delta_2)\delta_3(z_{C2}+v_{C2})^{\gamma\lambda}$$
$$+\delta_2(1-\delta_3)(z_{B2}+\min(0,z_{C2})+v_{C2})^{\gamma\lambda}+(1-\delta_2)(1-\delta_3)(\min(0,z_{C2})+$$
$$v_{C2})^{\gamma\lambda}\} - \alpha p_{A2}[\delta_2 G_{1H}(H_2,z_{B2},p_{B2}-p_{C2})+$$
$$(1-\delta_2)G_{1H}(H_2,0,p_{B2}-p_{C2})] = 0 \tag{A.42}$$

$$\frac{\partial f_2}{\partial z_{B2}} = \gamma\lambda z_{A2}^{(1-\gamma)\lambda}\{\delta_2\delta_3(z_{B2}+z_{C2}+c_{C2})^{\gamma\lambda-1}+\delta_2(1-\delta_3)(z_{B2}+\min(0,z_{C2})$$
$$+v_{C2})^{\gamma\lambda-1}\} - \alpha\delta_2[p_{C2}G_{1H}(H_2,z_{B2},p_{B2}-p_{C2})$$
$$-G_{1Z}(H_2,z_{B2},p_{B2}-p_{C2})] \le 0 \tag{A.43}$$

$$\frac{\partial f_2}{\partial z_{C2}} = \gamma\lambda z_{A2}^{(1-\gamma)\lambda}\{\delta_2\delta_3[(z_{B2}+z_{C2}+v_{C2})^{\gamma\lambda-1}-\alpha p_{C2}G_{1H}(H_2,p_{B2},p_{B2}-p_{C2})]$$
$$+(1-\delta_2)\delta_3[(z_{C2}+v_{C2})^{\gamma\lambda-1}-\alpha p_{C2}G_{1H}(H_2,0,p_{B2}-p_{C2})]$$
$$+\{\delta_2(1-\delta_3)[(z_{B2}+\min(0,z_{C2})+v_{C2})^{\gamma\lambda-1}-\alpha p_{C2}G_{1H}(H_2,z_{B2},p_{B2}-p_{C2})]$$
$$+(1-\delta_2)(1-\delta_3)[(\min(0,z_{C2})+v_{C2})^{\gamma\lambda-1}-\alpha p_{C2}G_{1H}(H_2,0,p_{B2}-p_{C2})]\}$$
$$\frac{\partial \min(0,z_{C2})}{\partial z_{C2}}\} = 0 \tag{A.44}$$

Note here that the total differentiation of the next period's maximum expected utility G_1 with respect to the order of the new durable goods z_{B2}, i.e.,

$$\frac{dG_1(\cdot)}{dz_{B2}} = G_{1Z}(\cdot) - p_{C2}G_{1H}(\cdot)$$

where the arguments of the functions are $H_2, z_{B2}, p_{B2}-p_{C2}$, gives the double effect of the purchase of a new durable good upon next period's maximum expected utility. The first term, G_{1Z}, gives the effect upon next period's expected utility that comes from the investment in durable goods. The second term, $-p_{C2}G_{1H}$, is always negative and shows the reduction in next period's maximum expected utility that resulted from the reduction in savings brought forward by the higher current consumption of services of new durable goods (achieved through a higher z_{B2}).

Let us examine closer the first order conditions of the problem. Suppose initially that z_{C2} is negative, i.e., $\min(0,z_{C2}) = z_{C2}$. Under this assumption, conditions (A.43) and (A.44) become

$$\begin{aligned}\gamma\lambda z_{A2}^{(1-\gamma)\lambda}(z_{B2}+z_{C2}+v_{C2})^{\gamma\lambda-1} &\leq \alpha[p_{C2}G_{1H}(H_2,z_{B2},p_{B2}-p_{C2}) \\ &\quad -G_{1Z}(H_2,z_{B2},p_{B2}-p_{C2})] \qquad \text{(A.45)}\end{aligned}$$

$$\begin{aligned}\gamma\lambda z_{A2}^{(1-\gamma)\lambda}[\delta_2(z_{B2}+z_{C2}+v_{C2})^{\gamma\lambda-1}&+(1-\delta_2)(z_{C2}+v_{C2})^{\gamma\lambda-1}] \\ &= \alpha p_{C2}[\delta_2 G_{1H}(H_2,z_{B2},p_{B2}-p_{C2}) \\ &\quad +(1-\delta_2)G_{1H}(H_2,0,p_{B2}-p_{C2})] \qquad \text{(A.46)}\end{aligned}$$

I can show, by contradiction, that condition (A.45) should hold with equality, i.e., the optimal order for new durable goods should be positive. Indeed, if $z_{B2} = 0$ then conditions (A.45) and (A.46) become correspondingly

$$\gamma\lambda z_{A2}^{(1-\gamma)\lambda}(z_{C2}+v_{C2})^{\gamma\lambda-1} < \alpha p_{C2}G_{1H}(H_2,0,p_{B2}-p_{C2})$$
$$-\alpha G_{1Z}(H_2,0,p_{B2}-p_{C2})$$
$$\gamma\lambda z_{A2}^{(1-\gamma)\lambda}(z_{C2}+v_{C2})^{\gamma\lambda-1} = \alpha p_{C2}G_{1H}(H_2,0,p_{B2}-p_{C2})$$

For $(p_{B2}-p_{C2}) < (p_{B2}-p_{C2})^u$, which we assumed true, equation (A.40) holds. Therefore, the above two conditions cannot hold simultaneously.

Next, assume that z^*_{C2} is non-negative, i.e., that $\min(0,z_{C2}) = 0$. Now conditions (A.43) and (A.44) for $z^*_{B2} = 0$, become

$$\gamma\lambda z_{A2}^{(1-\gamma)\lambda}[\delta_3(z_{C2}+v_{C2})^{\gamma\lambda-1}+(1-\delta_3)v_{C2}^{\gamma\lambda-1}]$$
$$< \alpha p_{C2}G_{1H}(H_2,0,p_{B2}-p_{C2})-\alpha G_{1Z}(H_2,0,p_{B2}-p_{C2}) \qquad (A.47)$$
$$\gamma\lambda z_{A2}^{(1-\gamma)\lambda}(z_{C2}+v_{C2})^{\gamma\lambda-1} = \alpha p_{C2}G_{1H}(H_2,0,p_{B2}-p_{C2}) \qquad (A.48)$$

Substitution of (A.48) in (A.47) yields

$$\gamma\lambda z_{A2}^{(1-\gamma)\lambda}(1-\delta_3)[v_{C2}^{\gamma\lambda-1}-(z_{C2}+v_{C2})^{\gamma\lambda-1}] + \alpha G_{1Z}(H_2,0,p_{B2}-p_{C2}) < 0,$$

which given (A.40) and $\gamma\lambda-1 <$, $z_{C2} \geq 0$, is not true. This implies that (A.47) has to hold with equality.

Thus, given that (A.45) and (A.47) must hold with equality, condition (A.43) has to hold with equality. In other words, as long as $p_{B2}-p_{C2}$ takes values inside its border values, the consumer will always order a positive amount of new durable goods.

Solution of the system (A.42)-(A.44), given that (A.43) holds with equality, will give the demand functions for z_{A2} and z_{B2} and the excess demand function for z_{C2}, as functions of the current prices, wealth and inventory, i.e., as functions of the state variables of the system. That is, the demand functions (excess demand for z_{C2}) will have the form

$$z^*_{A2} = z_{A2}(p_{A2},p_{B2},p_{C2},w_2,v_{C2}) \qquad (A.49)$$
$$z^*_{B2} = z_{B2}(p_{A2},p_{B2},p_{C2},w_2,v_{C2}) \qquad (A.50)$$
$$z^*_{C2} = z_{C2}(p_{A2},p_{B2},p_{C2},w_2,v_{C2}) \qquad (A.51)$$

Exact calculation of the above functions is very difficult, due partly to the fact that the exact specification of the functions $G_1(\cdot)$, $G_{1Z}(\cdot)$ and G_{1H} is very complicated. The following case will demonstrate it. Under the assumption that z_{C2} is negative the first order conditions (A.42)-(A.44) become

$$(1-\gamma)\lambda z_{A2}^{(1-\gamma)\lambda-1}[\delta_2(z_{B2}+z_{C2}+v_{C2})^{\gamma\lambda}+(1-\delta_2)(z_{C2}+v_{C2})^{\gamma\lambda}] \quad (A.52)$$

$$= \alpha p_{A2}[\delta_2 G_{1H}(H_2,z_{B2},p_{B2}-p_{C2})+(1-\delta_2)G_{1H}(H_2,0,p_{B2}-p_{C2})]$$

$$\gamma\lambda z_{A2}^{(1-\gamma)\lambda}(z_{B2}+z_{C2}+v_{C2})^{\gamma\lambda-1} = \alpha p_{C2}G_{1H}(H_2,z_{B2},p_{B2}-p_{C2}) \quad (A.53)$$

$$- \alpha G_{1Z}(H_2,z_{B2},p_{B2}-p_{C2})$$

$$\gamma\lambda z_{A2}^{(1-\gamma)\lambda}[\delta_2(z_{B2}+z_{C2}+v_{C2})^{\gamma\lambda-1}+(1-\delta_2)(z_{C2}+v_{C2})^{\gamma\lambda-1}] \quad (A.54)$$

$$= \alpha p_{C2}[\delta_2 G_{1H}(H_2,z_{B2},p_{B2}-p_{C2})+(1-\delta_2(G_{1H}(H_2,0,p_{B2}-p_{C2})]$$

Dividing (A.52) by (A.54) and solving with respect to z_{A2} yields

$$z_{A2} = \frac{(1-\gamma)p_{C2}[\delta_2(z_{B2}+z_{C2}+v_{C2})^{\gamma\lambda}+(1-\delta_2)(z_{C2}+v_{C2})^{\gamma\lambda}]}{\gamma p_{A2}[\delta_2(z_{B2}+z_{C2}+v_{C2})^{\gamma\lambda-1}+(1-\delta_2)(z_{C2}+v_{C2})^{\gamma\lambda-1}]}$$

Substituting back z_{A2} into (A.53) and (A.54) will result in a two equation system in two unknowns which can be solved in principle. It is evident that the exact solution depends upon the exact form of G_1 and its derivatives.

Some more results are now in order. First, examine again the consumers portfolio problem. Given the assumption that the new and used goods prices are such that the solution in the portfolio problem is interior, there will be a level of new goods order $\hat{z}_{B2}$ such that the portfolio returns will be maximized, i.e.,

$$G_{1Z}(H_2,\tilde{z}_{B2},p_{B2}-p_{C2}) = 0 \quad (A.55)$$

Assume now that the consumer maximizes its portfolio returns, i.e., orders $\hat{z}_{B2}$. Dividing (A.54) by (A.53), given that (A.55) holds, gives

$$\left(\frac{z_{C2}+v_{C2}}{\tilde{z}_{B2}+z_{C2}+v_{C2}}\right)^{\gamma\lambda-1} = \frac{G_{1H}(H_2,0,p_{B2}-p_{C2})}{G_{1H}(H_2,\hat{z}_{B2},p_{V2}-p_{C2})} \tag{A.56}$$

The LHS term is greater than one, except in the case where $\tilde{z}_{B2} = 0$, because $\gamma\lambda$-1 is negative. The RHS term is smaller than one, except in the case where $\hat{z}_{B2} = 0$, because the numerator, being constrained, is smaller than the denominator. Therefore, except in the case where $\hat{z}_{B2} = 0$ (which is excluded by the assumption that $p_{B2}-p_{C2}$ lies inside its border values), the above equation does not hold true. In other words, the optimal size of the new goods order cannot be determined by looking only at the consumer's portfolio problem (as in the fully executed orders case). Under positive probabilities of rationing, the consumer will order a quantity z^*_{B2} larger than the portfolio return maximizing quantity $\hat{z}_{B2}$. Thus, for $z_{B2} = z^*_{B2}$, equation (A.55) will be negative and equation (A.56) will have to be more complicated and will hold true. The larger order z^*_{B2} insures the consumer against the probability of rationing next period, by providing him with a larger inventory of used goods.

Next, I show that period 2's problem has also a critical inventory level. Assuming that $z_{C2} = 0$, the first order conditions become

$$(1-\gamma)\lambda z_{A2}^{(1-\gamma)\lambda-1}[\delta_2(z_{B2}+v_{C2})^{\gamma\lambda}+(1-\delta_2)v_{C2}^{\gamma\lambda}] \tag{A.57}$$
$$= \alpha p_{A2}[\delta_2 G_{1H}(H_2,z_{B2},p_{B2}-p_{C2})+(1-\delta_2)G_{1H}(H_2,0,p_{B2}-p_{C2})]$$

$$\gamma\lambda z_{A2}^{(1-\gamma)\lambda}(z_{B2}+v_{C2})^{\gamma\lambda-1} = \alpha p_{C2}G_{1H}(H_2,z_{B2},p_{B2}-p_{C2}) \tag{A.58}$$
$$- \alpha G_{1Z}(H_2,z_{B2},p_{B2}-p_{C2})$$

$$\gamma\lambda z_{A2}^{(1-\gamma)\lambda}[\delta_2(z_{B2}+v_{C2})^{\gamma\lambda-1}+(1-\delta_2)v_{C2}^{\gamma\lambda-1}] \tag{A.59}$$
$$= \alpha p_{C2}[\delta_2 G_{1H}(H_2,z_{B2},p_{B2}-p_{C2})+(1-\delta_2)G_{1H}(H_2,0,p_{B2}-p_{C2})]$$

This three equation system can be solved, in principle, with respect to z_{A2}, z_{B2} and v_{C2}. The optimal value for the inventory v^*_{C2}

will be a critical inventory level such that, when the actual inventory is above this critical level the consumer sell the excess amount, while if the actual inventory is below v^*_{C2}, the consumer orders the difference.

To prove the above argument let $v_{C2} + \varepsilon = v^*_{C2}$, where ε is positive in the case where the actual inventory is below the critical and negative in the opposite case.

Substitution of $v_{C2} + \varepsilon$ in (A.59) (which is the condition that determines v^*_{C2}) yields

$$\gamma\lambda z_{A2}^{(1-\gamma)\lambda}[\delta_2(z_{B2}+\varepsilon+v_{C2})^{\gamma\lambda-1}+(1-\delta_2)(\varepsilon+v_{C2})^{\gamma\lambda-1}] \tag{A.59'}$$
$$= \alpha p_{C2}[\delta_2 G_{1H}(H_2,z_{B2},p_{B2}-p_{C2})+(1-\delta_2)G_{1H}(H_2,0,p_{B2}-p_{C2})]$$

We can immediately see that condition (A.59') is the same as condition (A.54), for $z_{C2} = \varepsilon > 0$. Also assuming $z_{C2} \geq 0$, condition (A.44) reduces to a condition similar to (A.59') for $z_{C2} = \varepsilon \geq 0$. This proves that $z^*_{C2} = \varepsilon = v^*_{C2} - v_{C2}$.

Recapitulating, we showed that the first order conditions for the two-period horizon problem can be solved, in principle, to derive demand and excess demand functions depending upon the state variables of the system.

A number of extensions is possible. First, the durability of the durable good can be extended to more than two periods. This extension will result in an increase of the number of state and choice variables of the system. For example, while for the two-period durability case we examine here we have five state and three choice variables, extension to a three-period lasting good will increase the state variables to seven and the choice variables to four, i.e., we are going to have one more price, one more inventory, and more type of good to order. Apart from

that increase in variables, no other changes in the method of analysis are expected. A second extension has to do with the length of the horizon. We can extend the horizon to any length without altering any of the derived results. Finally, we should note that the model analyzed assumes that if the order is not delivered the order is lost for supplier. The case where the supplier backlogs his order can be examined. In the backlogging case we will have to assume that the order can be backlogged only for a finite number of periods and has given probabilities of been delivered each of these periods. Under this formulation each period the consumer faces certain probabilities of getting any of the orders he placed in the current or previous periods. All the previous undelivered orders should be included in the state variables of the system, which will result in a very complicated model. Nevertheless, the demand functions that will be derived from such a model will still depend upon current prices, wealth and inventories in addition to the undelivered orders.

The consumer's behavior analysis was undertaken in order to derive the form of the demand functions that the suppliers face when there is uncertainty in the market. As I said in the main text, the aggregation process from the individual functions derived here to the market functions that the suppliers face is not trivial, but due to the fact that there is no well developed theory of aggregation, I will follow the traditional route which suggests that the market demand functions will have the same form as the individual demand functions.

With respect to the properties (signs) of the demand functions, the fact that I was not able to get the exact forms of the functions (A.49)-(A.51), clearly suggests that derivation of their properties is extremely

difficult. Therefore, I will assume that the properties derived in the main text for the fully executed orders case carry over the present case where the probabilities of rationing are positive.

APPENDIX B

[1]The maximand G of the problem for $F = F^1$ can be written as

$$G(s,p,y,r,F^1) = w(s,p,y,r) + \alpha\int F^1(s-g_1(r,p)-u_1,r)d\Phi_1(u)d\Phi_3(r)$$

where

$$w(s,p,y,r) = pg_1(r,p) - c_1(s-y) - \int h(s-g_1(r,p)-u_1)d\Phi_1(u)$$

For $F = F^2$, the maximand becomes

$$G(s,p,r,F^2) = w(s,p,y,r) + \alpha\int F^2(s-g_1(r,p)-u_1,r)d\Phi_1(u)d\ {}_3(r)$$

Defining a metric $(F^1,F^2) = \sup_y|\int F^1(y,r)d\Phi-\int F^2(y,r)d\Phi|$ we can easily see that

$$|G(s,p,y,r,F^1) - G(s,p,y,r,F^2)| =$$

$$\alpha|\int F^1(.,r)d\Phi_1(u)d\Phi_3(r) - \int F^2(.,r)d\Phi(u)d\Phi_3(r)| \leq \alpha\rho(F^1,F^2).$$

This proves that Denardo's contraction assumption holds for the present problem.

Next, it is easy to see that for $F^1 \geq F^2$ it holds that

$$G(s,p,y,r,F^1) \geq G(s,p,y,r,F^2).$$

This shows that the monotonicity assumption is also satisfied.

REFERENCES

1. Bellman, Richard, Dynamic Programming. Princeton University Press, Princeton, New Jersey, 1957.

2. Benjamin, Daniel K., and Roger C. Kormendi, "The Interrelationship Between Markets of New and Used Durable Goods," Journal of Law and Economics, October 1974, 17:381-401.

3. Blinder, Alan S., "Inventories and Sticky Prices: More on the Microfoundations of Macroeconomics," American Economic Review, June 1983, 72:334-348.

4. Blinder, Alan S., "Can the Production Smoothing Model of Inventory Behavior Be Saved?" Working Paper No. 1257, National Bureau of Economic Research, January 1984.

5. Bradfield, James, "Monopoly, Product Durability and Second Hand Markets," Working Paper, Hamilton College, 1982.

6. Bulow, Jeremy, "Durable Goods Monopolists," Journal of Political Economy, April 1982, 90:314-332.

7. Coase, Ronald H., "Durability and Monopoloy," Journal of Law and Economics, April 1972, 15:143-149.

8. Denardo, Ernie V., "Contraction Mappings in the Theory Underlying Dynamic Programming," SIAM Review, April 1967, 9:165-177.

9. Hakansson, Nils H., "Optimal Investment and Consumption Strategies Under Risk for a Class of Utility Functions," Econonmetrica, September 1970, 38:587-607.

10. Iglehart, Donald L., "Capital Accumulation and Production for the Firm: Optimal Dynamic Policies," Management Science, November 1965, 12:193-205.

11. Liebowitz, S.J., "Durability, Market Structure, and New-Used Goods Models," American Economic Review, September 1982, 72:816-824.

12. Miller, Charles L., "Multiperiod Duopoly Under Uncertainty," Working Paper, University of Rochester, 1982.

13. Miller, H. Laurence, "On Killing Off the Market for Used Textbooks and the Relationship Between Markets for New and Secondhand Goods," Journal of Political Economy, May/June 1974, 82:612-619.

14. Parks, Richard W., "The Demand and Supply of Durable Goods and Durability," American Economic Review, March 1974, 64:37-55.

15. Reagan, Patrician B., "Inventory and Price Behavior," Review of Economic Studies, January 1982, 49:137-142.

16. Schmalensee, Richard, "Market Structure, Durability and Quality: A Selective Study," Economic Inquirey, April 1979, 17:171-196.

17. Stokey, Nancy L., "Rational Expectations and Durable Goods Pricing," Bell Journal of Economics, Spring 1981, 12:112-128.

18. Suslow, Valery Y., "Intertemporal Pricing Problems of a Durable Goods Monopolist," Working Paper No. 7, Stanford Workshop on Factor Markets, Stanford University, August 1982.

19. Swan, Peter L., "Market Structure and Technological Progress: The Influence of Monopoly on Product Innovation," Quarterly Journal of Economics, November 1970, 84:627-638.

20. Swan, Peter L., "Durability of Consumer Goods," American Economic Review, December 1970, 60:884-894.

21. Swan, Peter L., "ALCOA: The Influence of Recycling on Monopoly Power," Journal of Political Economy, February 1980, 88:76-99.

22. Thowsen, Gunnar T., "A Dynamic Nonstationary Inventory Problem for a Price/Quantity Setting Firm," Navel Research Logistics Quarterly, September 1975, 22:461-476.

23. Zabel, Edward, "Monopoly and Uncertainty," Review of Economic Studies, April 1970, 37:205-219.

24. Zabel, Edward, "Multiperiod Monopoly Under Uncertainty," Journal of Economic Theory, December 1972, 5:524-536.

25. Zabel, Edward, "Consumer Behavior Under Risk in Disequilibrium Trading," International Economic Review, June 1977, 18:323-343.

26. Zabel, Edward, "Competitive Price Adjustment Without Market Clearing," Econometrica, September 1981, 49:1201-1221.

27. Zabel, Edward, "Trading in a Monopolistic Market," Discussion Paper No. 50, Center for Econometrics and Decision Sciences, University of Florida, 1981.

BIOGRAPHICAL SKETCH

John Roufagalas was born June 19, 1956, in Athens, Greece. After graduating from high school in 1974, he entered the Pireaus Graduate School of Industrial Studies. He graduated during Fall 1978 with a B.S. in economics. He worked for a year as a research assistant in the Greek Center of Planning and Economic Development (KEPE).

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of doctor of philosophy.

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Professor of Economics

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August 1986

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